Plane Waves

- Most general kinds of waves are plane waves (sines, cosines, complex exponentials) – extend forever in space
- \( \psi_1(x,t) = \exp(i(k_1x-\omega_1t)) \)
- \( \psi_2(x,t) = \exp(i(k_2x-\omega_2t)) \)
- \( \psi_3(x,t) = \exp(i(k_3x-\omega_3t)) \)
- \( \psi_4(x,t) = \exp(i(k_4x-\omega_4t)) \)
- etc...

Different \( k \)'s correspond to different energies, since
\[
E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{\hbar^2}{2m\lambda^2} = \frac{\hbar^2k^2}{2m}
\]

Note: \( p = \hbar k \)
\[ E = \frac{hc}{\lambda} \ldots \]

A. ...is true for both photons and electrons.
B. ...is true for photons but not electrons.
C. ...is true for electrons but not photons.
D. ...is not true for photons or electrons.

\[ E = hf \text{ is always true} \] but \( f = \frac{c}{\lambda} \) only applies to light, so \( E = hf \neq \frac{hc}{\lambda} \) for electrons.

\( c = \text{speed of light!} \)
Announcements

- Reading the rest of chapter 6 and 7.3-7.5
- Homework 8 up today.
- Exams back on Friday.
- Extra office hours to discuss exam (will have normal ones too).
  - Wednesday 2:30-3:30
Plane Waves vs. Wave Packets

Plane Wave: $\Psi(x,t) = Ae^{i(kx-\omega t)}$

Wave Packet: $\Psi(x,t) = \sum_n A_n e^{i(knx-\omega n t)}$

For which type of wave are position $x$ and momentum $p$ most well-defined?

A. $p$ most well-defined for plane wave, $x$ most well-defined for wave packet.

B. $x$ most well-defined for plane wave, $p$ most well-defined for wave packet.

C. $p$ most well-defined for plane wave, $x$ equally well-defined for both.

D. $x$ most well-defined for wave packet, $p$ most well-defined for both.

E. $p$ and $x$ equally well-defined for both.
Plane Waves vs. Wave Packets

**Plane Wave:** \( \Psi(x,t) = Ae^{i(kx-\omega t)} \)

- Wavelength, momentum, energy: well-defined.
- Position: not defined. Amplitude is equal everywhere, so particle could be anywhere!

**Wave Packet:** \( \Psi(x,t) = \sum_n A_n e^{i(k_n x - \omega_n t)} \)

- \( \lambda, p, E \) not well-defined: made up of a bunch of different waves, each with a different \( \lambda, p, E \)
- \( x \) much better defined: amplitude only non-zero in small region of space, so particle can only be found there.
Plane Waves vs. Wave Packets

Plane Wave: \[ \psi(x,t) = A \exp(i(kx - \omega t)) \]

Wave Packet: \[ \psi(x,t) = \sum_n A_n \exp(i(k_n x - \omega_n t)) \]

Which one looks more like a particle?

• In real life, matter waves are more like wave packets. Mathematically, much easier to talk about plane waves, and we can always just add up solutions to get wave packet.

• Method of adding up sine waves to get another function (like wave packet) is called “Fourier Analysis.” You will explore it with simulation in the homework.
The Fourier transform of a function $f(x)$ is given by:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$$

where $k$ is the wave number and $x$ is the space variable. The inverse Fourier transform is:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{2\pi i k x} dk$$

The amplitude $A_n$ at each wave number $k_n$ is given by:

$$A_n = \frac{1}{\sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n k x} dx$$

The wave packet center is given by $k_0 = 37.7 \text{ rad/mm}$, and the wave packet width is $\sigma_k = 9.43 \text{ rad/mm}$.
“Uncertainty”... What does that mean?

We a series of electrons that satisfy some de Broglie/Born/Schroedinger wavefunction \( \Psi_x(x) = Ae^{-x^2} \) which looks like the graph below. If we measure many such electrons what is meant by their uncertainty?

a) If we measure the position of one of them it’s location at measurement can’t be determined to better than the “uncertainty”

b) If we measure the position of one of them it’s location will be within the “uncertainty” of 0.

c) If we measure the position of more than one of them they will have a distribution that has a width of the “uncertainty”

d) If we measure the position of one of them it is most likely to be with a width of the “uncertainty” from 0

e) c) and d) are both right.
Wave uncertainty

• In general waves are a bit “fuzzy”
• The exact location of a wave packet is somewhat uncertain. It has a distribution which has both an average value $u_0$ and an uncertainty $\Delta u = \sqrt{\langle (u - u_0)^2 \rangle}$: which is the standard deviation and $\langle \rangle$ means the average over that quantity.

• This is a general statement about all waves

• If a signal is made up of

\[ \Psi(u) = \sum_n A_n e^{i \nu_n u} \]

• Then there is an uncertainty relationship:

\[ \Delta u \Delta \nu \geq \frac{1}{2} \]

But you already know this, maybe, from radio signals.
Wave uncertainty

• This is true for radio and TV signals:
  \[ \sim \sin(\omega t) = \sin(2\pi ft) \]
  \[ \Delta f \Delta t \geq \frac{1}{2} \frac{1}{2\pi} \]

Why does this matter? Well what is needed to keep an audio signal?

What is the highest frequency in an audio signal?

a) 1 kHz
b) 5 kHz

c) 15 kHz

d) 20 kHz

e) 25 kHz

At best. Lowers with age.
Wave uncertainty

• This is true for radio and TV signals:
  \[ \sim \sin(\omega t) = \sin(2\pi f t) \]

  \[ \Delta f \Delta t \geq \frac{1}{2 \cdot 2\pi} \sim 0.1 \]

  This means to transmit audio with frequency \( f \text{ < } 20 \text{ kHz}, \) you have signals which change with timescales of \( \Delta t \sim 1/20\text{kHz}. \)

  Need bandwidth \( \Delta f \) of < 20 kHz. (for better bad fidelity need 2kHz, for perfect fidelity need 20kHz)

  Actually use 200 kHz so extra space for name of song, gps info, stock information, lotsa stuff.
What about matter waves?

- Matter wave packets are made up of
  \[ \Psi(x) = \sum_n A_n e^{i k_n x} \]

- Then there is an uncertainty relationship:
  \[ \Delta k_x \Delta x \geq \frac{1}{2} \]

Remember: \( k = \frac{2\pi}{\lambda} \) and \( \lambda = \frac{\hbar}{p} \)

\[ \Delta p_x \Delta x \geq \frac{\hbar}{2} \quad \text{or} \quad \Delta p_y \Delta y \geq \frac{\hbar}{2} \quad \text{and} \quad \Delta p_z \Delta z \geq \frac{\hbar}{2} \]
Heisenberg Uncertainty Principle

• In math: $\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$

• In words: Position and momentum cannot both be determined completely precisely. The more precisely one is determined, the less precisely the other is determined.

• Should really be called “Heisenberg Indeterminacy Principle.”

• This is weird if you think about particles, not very weird if you think about waves.
Heisenberg Uncertainty Principle

\[ \Delta x \] small \( \Delta p \) – only one wavelength

\[ \Delta x \] medium \( \Delta p \) – wave packet made of several waves

\[ \Delta x \] large \( \Delta p \) – wave packet made of lots of waves
A slightly different scenario:

Plane-wave propagating in x-direction.
\( \Delta y: \) very large \( \rightarrow \Delta p_y: \) very small

Restriction in y directly:
Small \( \Delta y \rightarrow \) large \( \Delta p_y \)
\( \rightarrow \) wave spreads out strongly in y direction!

Weak restriction in y:
medium \( \Delta y \)
\( \rightarrow \) medium \( \Delta p_y \)
\( \rightarrow \) wave spreads out weakly in y direction!

\[ \Delta y \Delta p_y \geq \hbar/2 \]
What about time dependence?

- Matter wave packets are really are made up of

\[ \Psi(x) = \sum_m \sum_n A_n e^{i(k_n x - \omega_m t)} \]

- They can be made up of different frequencies in time or wave-vectors (k or frequency in space) at the same time so:

\[ \Delta \omega \Delta t \geq \frac{1}{2} \]

Remember for light: \( E = hf = \frac{h\omega}{2\pi} = \hbar \omega \)

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

This means the uncertainty in time and energy are related. So a state with well defined energy has a long “lifetime”
Review ideas from matter waves:
Electron and other matter particles have wave properties.
See electron interference
If not looking, then electrons are waves ... like wave of fluffy cloud.
As soon as we look for an electron, they are like hard balls.
Each electron goes through both slits ... even though it has mass.
(SEEMS TOTALLY WEIRD! Because different than our experience. Size scale of things we perceive)

If all you know is fish, how do you describe a moose?

Electrons & other particles described by wave functions ($\Psi$)
Not deterministic but probabilistic

Physical meaning is in $|\Psi|^2 = \Psi^*\Psi$
$|\Psi|^2$ tells us about the probability of finding electron in various places. $|\Psi|^2$ is always real, $|\Psi|^2$ is what we measure
Work towards finding an equation that describes/predicts the probability wave for matter in any situation.

Solving this (differential) equation will give incredible insight to the inner workings of nature and technology.

Look at general aspects of wave equations … apply to classical and quantum wave equations.
The Schrödinger Equation

\[
\frac{-\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}
\]

Once at the end of a colloquium Felix Bloch heard Debye saying something like: “Schrödinger, you are not working right now on very important problems...why don’t you tell us some time about that thesis of deBroglie, which seems to have attracted some attention?” So, in one of the next colloquia, Schrödinger gave a beautifully clear account of how deBroglie associated a wave with a particle, and how he could obtain the quantization rules...by demanding that an integer number of waves should be fitted along a stationary orbit. When he had finished, Debye casually remarked that he thought this way of talking was rather childish...To deal properly with waves, one had to have a wave equation.
Schroedinger equation development--
new approach to physics

Old approach- understand physical system well, reason out equations that must describe it.

New approach-- see what experiment shows but not know why. Write down equation that has sensible basic math properties. See if solutions of these equations match the experiment.

Schrödinger starting point: what do we know about classical waves (radio, violin string)?

What aspects of electron wave equation need to be similar, and what need to be different from those wave equations?
Review: classical wave equations

Electromagnetic waves:

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

$c = \text{speed of light}$

Solutions: $E(x,t)$

Magnitude is non-spatial:

$= \text{Strength of Electric field}$

Vibrations on a string:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

$v = \text{speed of wave}$

Solutions: $y(x,t)$

Magnitude is spatial:

$= \text{Vertical displacement of String}$
In DiffEq class, learn lots of algorithms for solving DiffEq’s. In Physics, only ~8 differential equations you ever need to solve, solutions are known, just guess them and plug them in.

How to solve a differential equation in physics:
1) Guess functional form for solution
2) Make sure functional form satisfies Diff EQ
   (find any constraints on constants)
   1 derivative: need 1 soln $\rightarrow f(x,t)=f_1$
   2 derivatives: need 2 soln $\rightarrow f(x,t) = f_1 + f_2$
3) Apply all boundary conditions
   (find any constraints on constants)

(You did this in HW6)
Which of the following functional forms works as a possible solution to this differential equation?

A. \( y(x, t) = Ax^2t^2 \),
B. \( y(x, t) = Asin(Bx) \)
C. \( y(x,t) = A\cos(Bx)\sin(Ct) \)
D. Both, B&C work!
E. None or some other combo

Test your idea. Does it satisfy Diff EQ?  (Just like HW6 prob 2)

Answer is C. Only: \( y(x,t)=A\cos(Bx)\sin(Ct) \)
1) Guess functional form for solution

New guess: \( y(x,t) = A\cos(Bx)\sin(Ct) \)

LHS: \( \frac{\partial^2 y}{\partial x^2} = -AB^2 \sin(Bx) \)

RHS: \( \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0 \)

\(-AB^2 \sin(Bx) = 0\)
\(\sin(Bx) = 0\)

Not OK! \( x \) is a variable. There are many values of \( x \) for which this is not true!

OK! \( B \) and \( C \) are constants. Constrain them so satisfy this.

\( B^2 = \frac{C^2}{v^2} \)
\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

Functional form of solution:
\[y(x,t) = A\sin(kx)\cos(\omega t) + B\cos(kx)\sin(\omega t)\]

Boundary conditions?
\[y(x,t) = 0 \quad \text{at} \quad x=0\]

At \(x=0\):
\[y(x=0,t) = B\sin(\omega t) = 0 \quad \Rightarrow \text{only works if} \quad B=0\]

Is that it? Does this eqn. describe the oscillation of a guitar string? What is \(k\)?
\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

\(y(x,t) = 0\) at \(x=L\)

Is there another boundary condition?

\(y(x,t) = Asin(kx)cos(\omega t)\)

\(y(x,t) = 0\) at \(x=L\)

At \(x=L\):
\[y = Asin(kL)cos(\omega t) = 0\]
\[\Rightarrow \sin(kL)=0\]
\[\Rightarrow kL = n\pi \quad (n=1,2,3, \ldots)\]
\[\Rightarrow k=n\pi/L\]

\[y(x,t) = Asin(n\pi x/L)cos(\omega t)\]

Quantization of \(k\) … quantization of \(\lambda\) and \(\omega\)
With Wave on Violin String:

Find: Only certain values of \( k, \lambda, \omega \rightarrow \) i.e., the frequencies of the string are quantized.

Same as for electromagnetic wave in microwave oven:

Exactly same for Electrons in atoms:

Find: Quantization of electrons energies (wavelengths) …

→ from boundary conditions for solutions to Schrodinger’s Equation.
A) The model we talked about is inconsistent with the ability to “tune” a violin. But violins are classical objects anyway, so no problem!

B) You can discuss the violin in terms of quantized standing waves yet still tune a violin. What gives?

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \rightarrow k = n\pi/L, \text{ and } \lambda = \frac{2L}{n}
\]

Can tune!!

But:

\[
v = \frac{\lambda}{T} = \frac{\omega}{k} \rightarrow f = \frac{v}{\lambda} \rightarrow v = \sqrt{\frac{T}{\rho}}
\]

\(\rho=\text{mass density}\)

\(T = \text{tension}\)