Conservation of Angular Momentum = "Spin"

We can assign a direction to the angular velocity:
direction of $\omega = $ direction of axis + right hand rule
(with right hand, curl fingers in direction of rotation, thumb points
in direction of $\omega$.) The direction of $\omega$ is abstract. There is nothing
actually moving in the direction of the $\omega$ vector. We want to choose the direction of $\omega$ so that
this gives us information about how the object is spinning. When a wheel is spinning on an axis,
there is no one direction of motion (different parts of the wheel are moving in different
directions). But there is always only one direction of the axis of rotation. So by giving both the
direction of the axis (direction of $\omega$) and how fast the object is spinning around that axis
(magnitude of $\vec{\omega} = |\vec{\omega}| = |d\theta|/dt$), we are describing its spinning motion exactly.

Similarly, we can assign a direction to angular acceleration $\alpha$:

$$\ddot{\alpha} = \frac{d\dot{\omega}}{dt} \approx \frac{\Delta \omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

Direction of $\alpha$ is direction of $\Delta \omega$, not the direction of $\omega$ (like
direction of $a$ is direction of $\Delta v$, not the direction of $v$)

We can also assign a direction to torque with the “cross product”.

The cross-product of two vectors $A$ and $B$ is a third vector $A \times B = C$ defined like this: The
magnitude is $|A \times B| = AB \sin \theta$. The direction of $C = A \times B$ is the direction perpendicular to
the plane defined by the vectors $A$ and $B$, plus right-hand-rule. (Curl fingers from first vector $A$
to second vector $B$, thumb points in direction of $A \times B$

\begin{align*}
\hat{i} \times \hat{j} &= \hat{k} & \hat{i} \times \hat{i} &= 0 \\
\hat{j} \times \hat{k} &= \hat{i} & \hat{j} \times \hat{j} &= 0 \quad \text{etc.}
\end{align*}
Vector torque defined as \[ \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \]

The vector torque \( \mathbf{\tau} \) is defined with respect to an origin (which is usually, but not always, the axis of rotation). So, if you change the origin, you change the torque (since changing the origin changes the position vector \( \mathbf{r} \)).

With these definitions of vector angular acceleration and vector torque, the fixed-axis equation \( \mathbf{\tau} = I \mathbf{\alpha} \) becomes

\[ \mathbf{\tau} = I \ddot{\mathbf{\omega}} = I \frac{d\mathbf{\omega}}{dt} \quad (\text{like } \mathbf{F} = m \ddot{\mathbf{v}} = m \frac{d\mathbf{v}}{dt}) \]

Now, a new concept: angular momentum \( \mathbf{L} = \text{"spin"} \). Angular momentum, a vector, is the rotational analogue of linear momentum. So, based on our analogy between translation and rotation, we expect \( \mathbf{L} = I \mathbf{\omega} \quad (\text{like } \mathbf{p} = m \mathbf{v}) \). Note that this equation implies that the direction of \( \mathbf{L} \) is the direction of \( \mathbf{\omega} \).

**Definition of angular momentum** of a particle with momentum \( \mathbf{p} = m \mathbf{v} \) at position \( \mathbf{r} \) relative to an origin is \[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \]

Like torque \( \mathbf{\tau} \), the angular momentum \( \mathbf{L} \) is defined w.r.t. an origin, often the axis of rotation.

We now show that the total angular momentum of a object spinning about a fixed axis is \( \mathbf{L} = I \mathbf{\omega} \). Consider a object spinning about an axis pointing along the +z direction. We place the origin at the axis.
If something has a big moment of inertia \( I \) and is spinning fast (big \( \omega \)), then it has a big "spin", big angular momentum. Angular momentum is a very useful concept, because angular momentum is conserved.

Important fact: the angular momentum of a object spinning about a symmetry axis that passes through the center of mass is given by \( \mathbf{L} = I_{cm} \mathbf{\omega} \), independent of the location of the origin; that is, even if the origin is chosen to be outside the spinning object, the angular momentum has the same value as if the origin was chosen to be at the axis. (Proof given in appendix).

**Conservation of Angular Momentum:** If a system is isolated from external torques, then its total angular momentum \( \mathbf{L} \) is constant.

\[
\tau_{ext} = 0 \Rightarrow L_{tot} = I \mathbf{\omega} = \text{constant} \quad (\text{like } F_{ext} = 0 \Rightarrow p_{tot} = \text{constant})
\]

Here is a proof of conservation of angular momentum:

First, we argue that \( \mathbf{\tau}_{net} = \frac{d\mathbf{L}}{dt} \) (this is like \( \mathbf{F}_{net} = \frac{d\mathbf{p}}{dt} \)):

\[
\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \left( \sum_i \mathbf{r}_i \times \mathbf{p}_i \right) = \sum_i \left( \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \right).
\]

Now, the first term in the last expression is zero:

\[
\sum_i \left( \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i \right) = \sum_i (\mathbf{\dot{v}}_i \times m_i \mathbf{\dot{v}}_i) = \sum_i m_i (\mathbf{\dot{v}}_i \times \mathbf{\dot{v}}_i) = 0 \text{, since any vector crossed into itself is zero. So, we have } \frac{d\mathbf{L}}{dt} = \sum_i \left( \mathbf{\dot{r}}_i \times \mathbf{\dot{p}}_i \right) = \sum_i (\mathbf{\dot{r}}_i \times \mathbf{\dot{r}}_i) \text{ (since } \mathbf{\dot{F}}_{net} = \frac{d\mathbf{p}}{dt}) \text{. Finally, } \sum_i (\mathbf{\dot{r}}_i \times \mathbf{\dot{r}}_i) = \sum_i \mathbf{\ddot{r}}_i = \mathbf{\tau}_{net}, \text{ so we have } \frac{d\mathbf{L}}{dt} = \mathbf{\tau}_{net}.
\]
So now we have, \( \tau_{\text{net}} = \frac{\Delta L}{\Delta t} \Rightarrow \) if \( \tau_{\text{net}} = 0 \), then \( \frac{\Delta L}{\Delta t} = 0 \Rightarrow L = \text{constant} \). Done.

It turns out that only 4 things are conserved:
- Energy
- Linear momentum \( p \)
- Angular momentum \( L \)
- Charge \( q \)

Conservation of Angular Momentum is very useful for analyzing the motion of spinning objects isolated from external torques — like a skater or a spinning star.

If \( \tau_{\text{ext}} = 0 \), then \( L = I \omega = \text{constant} \). If \( I \) decreases, \( \omega \) must increase, to keep \( L = \text{constant} \).

**Example: spinning skater.**

\[
I_i \omega_i = I_f \omega_f
\]

( \( I \) big, \( \omega \) small )  
( \( I \) small, \( \omega \) big )

**Example: rotation of collapsing star.** A star shines by converting hydrogen (H) into helium (He) in a nuclear reaction. When the H is used up, the nuclear fire stops, and gravity causes the star to collapse inward.

As the star collapses (pulls its arms in), the star rotates faster and faster. Star's radius can get much smaller: \( R_i \approx 1 \) million miles \( \rightarrow R_f \approx 30 \) miles
\[ I_i \omega_i = I_f \omega_f \quad (\text{Sphere I} = \frac{2}{3} M R^2) \]

\[
\left(\frac{2}{3} M R_i^2\right) \omega_i = \left(\frac{2}{3} M R_f^2\right) \omega_f
\]

\[
R_i^2 \omega_i = R_f^2 \omega_f
\]

\[
\frac{R_i^2}{R_f^2} = \frac{\omega_f}{\omega_i} = \frac{T_i}{T_f} \quad \text{(using } \omega = 2\pi f = \frac{2\pi}{T})
\]

If \( R_i >> R_f \), then \( T_i >> T_f \).

The sun rotates once every 27 days. "Neutron stars" with diameter of about 30 miles typically rotate 100 times per second.

Let's review the correspondence between translational and rotational motion:

<table>
<thead>
<tr>
<th>Translation</th>
<th>Rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>( v = \frac{\Delta x}{\Delta t} )</td>
<td>( \omega = \frac{\Delta \theta}{\Delta t} )</td>
</tr>
<tr>
<td>( a = \frac{\Delta v}{\Delta t} )</td>
<td>( \alpha = \frac{\Delta \omega}{\Delta t} )</td>
</tr>
<tr>
<td>( F )</td>
<td>( \tau = r F_\perp )</td>
</tr>
<tr>
<td>( M )</td>
<td>( I = \sum m r^2 )</td>
</tr>
<tr>
<td>( F_{\text{net}} = M a )</td>
<td>( \tau_{\text{net}} = I \alpha )</td>
</tr>
<tr>
<td>( KE_{\text{trans}} = (1/2) M v^2 )</td>
<td>( KE_{\text{rot}} = (1/2) I \omega^2 )</td>
</tr>
<tr>
<td>( p = m v )</td>
<td>( L = I \omega )</td>
</tr>
<tr>
<td>( F_{\text{net}} = \Delta p / \Delta t )</td>
<td>( \tau_{\text{net}} = \Delta L / \Delta t )</td>
</tr>
</tbody>
</table>

If \( F_{\text{ext}} = 0 \), \( p_{\text{tot}} \) = constant \( \leftrightarrow \) If \( \tau_{\text{ext}} = 0 \), \( L_{\text{tot}} \) = constant
Appendix. The angular momentum \( \mathbf{L} = \mathbf{I} \mathbf{\omega} \) of an object spinning about a symmetry axis is independent of the location of the origin. You can place the origin anywhere and you will get the same \( \mathbf{L} \) that you get when you place the origin at the center-of-mass of the spinning object.

Proof: Consider an object spinning about a stationary symmetry axis:

\[
\mathbf{\hat{r}} = \mathbf{\hat{R}} + \mathbf{\hat{r}}'
\]

position of CM position relative to CM

\[
\mathbf{P}_{\text{tot}} = 0 \quad \hat{\mathbf{L}} \text{ relative to origin at CM}
\]

More generally, if the center-of-mass \( \mathbf{R} \) is moving with velocity \( \mathbf{V}_{\text{CM}} \neq 0 \), then

\[
\hat{\mathbf{L}}_{\text{tot}} = \mathbf{\hat{R}} \times \mathbf{M} \mathbf{V}_{\text{cm}} + \sum_i (\mathbf{\hat{r}}' \mathbf{p}_i + m_i \mathbf{\hat{r}}' \mathbf{v}_i)
\]

\[
= \hat{\mathbf{L}}_{\text{motion of CM}} + \hat{\mathbf{L}}_{\text{motion relative to CM}}
\]

\[
= \hat{\mathbf{L}}_{\text{orbital}} + \hat{\mathbf{L}}_{\text{spin}}
\]