Mathematical Interlude: Vectors

Consider this vector diagram:

Which of these equations correctly describes the diagram?

A) \( \vec{A} + \vec{B} + \vec{C} + \vec{D} = 0 \)
B) \( \vec{A} + \vec{B} + \vec{C} = \vec{D} \)
C) \( \vec{A} + \vec{B} = \vec{C} + \vec{D} \)
D) \( \vec{A} + \vec{D} = \vec{B} + \vec{C} \)
E) None of these is correct.

Answer: \( \vec{A} + \vec{D} = \vec{B} + \vec{C} \)
Mathematical Interlude: Vectors

Expressing motion in 2D:
• Reviewed vectors – length (magnitude) and direction
• Expressing vectors in component form
• Finding length of a vector
• Finding components of a vector – requires trig
• Finding the direction from the components – more trig
• Review of trigonometry
• Addition & subtraction of vectors: by components & graphically
Three vectors \( \vec{A}, \vec{B}, \) and \( \vec{C} \) are as shown. Which vector is:

\[
\vec{S} = \vec{A} + \vec{B} - \vec{C}
\]

\( \vec{A} = (0, -2) \)
\( \vec{B} = (-2, -2) \)
\( -\vec{C} = (3, 0) \)
\( \vec{S} = (1, -4) \)

(A) (B) (C) (D)

(E) None of these.
Another way to express a vector is with “unit vectors”:

\[ \hat{i} = \hat{x} = \text{vector of length 1 in } x\text{-direction} = (1,0) \]
\[ \hat{j} = \hat{y} = \text{vector of length 1 in } y\text{-direction} = (0,1) \]

\[ \vec{A} = A_x \hat{x} + A_y \hat{y} \]
\[ = A_x \hat{i} + A_y \hat{j} \]
\[ = (A_x, A_y) \]
Mathematical Interlude: Vectors

- $\vec{A} = \hat{j}$

- $\vec{B} = 3\hat{i} + \hat{j} - 5\hat{j}$

- $\vec{C} = \frac{\vec{A}}{3}$

- $\vec{C} = \vec{A} + \hat{i}$

- $\vec{C} = 4 - \hat{i}$

How many of these equations do not make sense?

A) 0 (All make sense)  B) 1  C) 2  D) 3  E) 4

You can’t add a vector to a number.
Kinematics in 2D

Represent the position, velocity, and acceleration of an object:

1. Choose coordinates

2. Define the position vector:

   $\vec{r}$ points from the origin to the object’s location

   components: $\vec{r} = (x, y)$

   magnitude & direction:

   $$|\vec{r}| = \sqrt{x^2 + y^2}$$

   $$\theta = \tan^{-1} \left( \frac{y}{x} \right)$$
Kinematics in 2D

Represent the position, velocity, and acceleration of an object:

3. Define displacement
   \[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 \]

4. Define velocity:
   a) constant velocity:
   \[ \vec{v} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{\Delta \vec{r}}{\Delta t} \]

**Note:**
1. Velocity is a vector.
2. Its direction is determined entirely by displacement.
The position vector of a particle moving at constant velocity is shown below at two different times, an earlier time $t_1$ and a later time $t_2$. Which arrow shows the **direction** of the velocity vector?

(A) \( \vec{R}_1 \)  
(B) \( \vec{R}_2 \)  
(C)  
(D)  
(E) None of these!
Note: The velocity vector is in the direction of the displacement.
Kinematics in 2D

Represent the position, velocity, and acceleration of an object:

3. Define displacement
\[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 \]

4. Define velocity:
   a) constant velocity:
   \[ \vec{v} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{\Delta \vec{r}}{\Delta t} \]
   b) instantaneous velocity:
   \[ \vec{v} = (v_x, v_y) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d}{dt} (x, y) = (dx / dt, dy / dt) \]
Kinematics in 2D

Represent the position, velocity, and acceleration of an object:

3. Define displacement

\[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 \]

4. Define velocity:

   a) constant velocity:

   \[
   \vec{v} = \frac{\text{displacement}}{\text{elapsed time}} = \frac{\Delta \vec{r}}{\Delta t}
   \]

   b) instantaneous velocity:

   Reduces to 2 scalar eqns:

   \[
   v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt}
   \]
Kinematics in 2D

Represent the position, velocity, and acceleration of an object:

4. Define acceleration:
   a) constant acceleration:
   \[ \vec{a} = \frac{\Delta \vec{v}}{\Delta t} \]
   \[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 \]
   \[ \vec{a} = \left( a_x, a_y \right) = \frac{d\vec{v}}{dt} = \left( \frac{dv_x}{dt}, \frac{dv_y}{dt} \right) \]
   b) instantaneous acceleration:
Kinematics in 2D

Represent the position, velocity, and acceleration of an object:

4. Define acceleration:
   a) constant acceleration:
      \[ \vec{a} = \frac{\Delta \vec{v}}{\Delta t} \]
   b) instantaneous acceleration:
      \[ a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dt} \]

Reduces to 2 scalar eqns:
Which diagram below correctly depicts the equation \( \vec{V}_1 + \Delta \vec{V} = \vec{V}_2 \)?

D) all three (A, B, C) are correct  
E) None (A, B, C) are correct
**Kinematics in 2D**

Represent the position, velocity, and acceleration of an object:

<table>
<thead>
<tr>
<th>vector</th>
<th>$x$-component</th>
<th>$y$-component</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{r}$</td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>$\vec{v}$</td>
<td>$v_x = \frac{dx}{dt}$</td>
<td>$v_y = \frac{dy}{dt}$</td>
</tr>
<tr>
<td>$\vec{a}$</td>
<td>$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$</td>
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</tr>
</tbody>
</table>
Projectile Motion in 2D

PhET Sim on projectile motion:
http://phet.colorado.edu/en/simulation/projectile-motion
Projectile Motion in 2D

1. Acceleration is purely vertical.

2. Object speeds up vertically but not horizontally. It simply glides along horizontally.

3. Horizontal and vertical motions can be considered entirely separately from one another.
Projectile Motion in 2D

If an object is launched at an initial angle of $\theta_0$ with the horizontal, the analysis is similar except that the initial velocity has a vertical component.
Projectile Motion in 2D

Vertical y-direction

\[ a_y = -g \]
\[ y_0 \]
\[ v_{0y} \]

Horizontal x-direction

\[ a_x = 0 \]
\[ x_0 \]
\[ v_{0x} \]
Projectile Motion in 2D

Assume you have an object whose initial position is on the y-axis on the top of a hill of height $h$ above the origin. What’s $x_0$ and $y_0$?

A) $x_0 = h$, $y_0 = h$
B) $x_0 = 0$, $y_0 = h$
C) $x_0 = 0$, $y_0 = 0$
D) $x_0 = h$, $y_0 = 0$
Consider an object fired at an angle $\theta$ from the top of a hill to the right with a speed of $v_0$. What will the initial velocities in the x and y directions be?

A) $v_{x_0} = v_0 \cos \theta$, $v_{y_0} = 0$

B) $v_{x_0} = v_0 \sin \theta$, $v_{y_0} = 0$

C) $v_{x_0} = v_0 \cos \theta$, $v_{y_0} = v_0 \sin \theta$

D) $v_{x_0} = v_0 \sin \theta$, $v_{y_0} = v_0 \cos \theta$

E) $v_{x_0} = -v$, $v_{y_0} = v$
Kinematic Equations in 2D

Vertical y-direction

\[ a_y = -g \]
\[ v_{0y} = v_0 \sin \theta \]

Horizontal x-direction

\[ a_x = 0 \]
\[ v_{0x} = v_0 \cos \theta \]