Multipole Expansion
(Griffiths Chapter 3)

**Multipole Expansion**

**Question 1. Multipole moment of charged wire**
CALCULATION; EXPANSION (U. Nauenberg, HW3, solutions available)
A charge $Q$ is uniformly distributed along a straight non-conducting wire
of length “a” with a charge/unit length = “$\lambda$”.
(a) Calculate the potential $\phi$ by integration using the origin of your coordinate system the middle of the wire.
(b) Calculate the monopole, dipole, and quadrupole terms in the multipole expansion of the potential $\phi$ done in class.
(c) Expand the potential found in part (a) up to terms in $1/r^3$ and compare with the results in part (b).

**Question 2. Quadrupole and V for four charges**
MULTIPOLE; CALCULATION (deGrand)
Find the quadrupole moment and approximate $V(r)$ for the charge distribution of four point charges arranged in a square: $q = +1$ at coordinates $(x,y,z) = (-a,a,0)$ and $(a,-a,0)$, and $q = -1$ at $(a,a,0)$ and $(-a,-a,0)$.

**Question 3. Point charges**
You have four point charges. Their location and charges in Cartesian coordinates are:
A charge $-q$ located at $(a,0,0)$, another charge $-q$ located at $(-a,0,0)$, a third charge $+3q$ located at $(0,0,b)$, and finally a fourth charge $-q$ located at $(0,0,-b)$.
Find a simple approximate formula for the $V(r,\theta)$ (i.e. in spherical coordinates!) valid at points far from the origin. ("Simple" means only the first non-zero term is needed!)
ii) Find a simple approximate expression for the electric field valid at points far from the origin. (Again, express your answer in spherical coordinates, so we want $\vec{E}(r,\theta) = E_r(r,\theta)\hat{r} + E_\theta(r,\theta)\hat{\theta}$, and you should figure out what $E_r$ and $E_\theta$ are.)
Sketch this $E$ field. (Don't worry about what happens near the origin, I just want a sketch of the simple approximation)

**Assigned in SP08 (average score: i) 3.9, ii) 3.76)**

**Assigned in FA08**
Instructor notes: Some students add the dipole moments like scalars, not vectorially. Graphing seems to be difficult for students, and there is some evidence of cheating on this problem (graphs that are right despite differences in the calculated $E$ field).
**Question 4. Potential of finite rod**

MULTIPOLE; CALCULATION (deGrand)

You may recall that we computed the radial component of electric field from a rod length $L$ and uniform charge per unit length $\lambda$ a distance $r$ from its center along the axis of the rod (call it $\theta = 0$ in the picture) and perpendicular to it ($\theta = \pi/2$). We found the leading connection to Coulombic behavior to be

$$E_r = \frac{1}{4\pi \varepsilon_0} \frac{\lambda L^2}{4r^4}$$

And

$$E_r = -\frac{1}{4\pi \varepsilon_0} \frac{\lambda L^2}{8r^4}$$

respectively. Obtain these results directly by finding the potential using the Legendre expansion and taking appropriate derivatives. (Note: you can also find the potential for all $(r,\theta)$ by direct integration, but I do not recommend trying to Taylor expand it!)

**Question 5. Potential of charged sphere with cap**

CALCULATION; LIMITING FORMS (Jackson, Classical Electrodynamics, 2nd Ed., Problem 3.2)

A spherical surface of radius $R$ has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the bounding cone $\theta = \alpha$. a. Before we get started here, show the following recursion relation for Legendre polynomials to be true:

$$\frac{dP_{l-1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} - (2l + 1)P_l(x) = 0$$

(for $l > 0$).

This can be obtained by considering the Rodriques formula (Griffiths, (3.62)).

b. Show that the potential inside the spherical surface can be expressed as

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} - (2l + 1)P_l(x) = 0$$

where, for $l = 0$, we should substitute $P_0(x) = 1$.

c. What is the potential outside the sphere?

d. Find the magnitude and the direction of the electric field at the origin.

e. Discuss the limiting forms of the potential (b,c) and the electric field (d) as the spherical cap becomes either (1) very small or (2) so large that the entire remaining charged area becomes a very small cap at the south pole.

**Question 6. Quadrupole moment of nuclei**

REAL-WORLD, CALCULATION (Berkeley, Stamper-Kurn)

Many nuclei have electric quadrupole moments. The quadrupole moment of a given state of a given nucleus is typically quantified as $Q = Q_{33}/e$ where $Q_{ij}$ is defined as in Griffiths’ Problem 3.45, and the index “3” refers to the $\hat{z}$ axis. Now, consider a nucleus centered at the origin finding itself in a cylindrically symmetric electric field with a gradient $\partial E_z / \partial z$ along the $\hat{z}$ axis at the position of the nucleus.

a. Show that the inhomogeneous field modifies the energy of the quadrupole by
For this, you might want to find a suitable form of the electrostatic potential near the position of the nucleus. Be sure that this potential satisfies the Laplace equation. You should let $V=0$ at the origin.

b. Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal (egg-shaped) volume of semimajor axis $a$ and semimajor axis $b$. Calculate the quadrupole moment of such a nucleus assuming that the total charge is $Ze$.

Given that the 153 Eu ($Z = 63$) nucleus has a quadrupole moment of $Q = 2.5 \times 10^{-25}$ cm$^2$ and a mean radius $R = (a+b)/2 = 7 \times 10^{-13}$ cm, determine the fractional difference in radius $(a - b)/R$.

**Question 7. Spherical charge distribution**

Griffiths derives (on page 142-144) the exact potential $V(r,\theta)$ everywhere outside a spherical shell of radius $R$ which has a surface charge distribution $\sigma(R,\theta) = k\cos\theta$

i) Calculate the dipole moment of this object, and also the total charge on this object.

ii) Use the methods of "the multipole expansion" (Griffiths p. 148) to find an approximate form for the potential far from the sphere. You can stop with the leading nonzero term. Now compare with Griffiths exact formula from the earlier example (on p.144). What does this tell you about the quadrupole moment (and higher moments) of this surface charge distribution?

**Assigned in SP08 (average score: i) 7.9, ii) 3.62)**

Instructor notes: Almost all students fail to argue by symmetry that $px=py=0$. Most fail to show interest in the fact that this distribution gives a pure dipole (part ii), suggesting they are not deeply considering the physics.

**Question 8. Multipole moments of charged cylinder**

CALCULATION, NEW SITUATION (G. Gwinn, UC Santa Barbara)

Consider a cylinder of radius $a$, and of height $h$. The surface of the cylinder is covered with surface charge. The end caps of the cylinder have surface charge $\sigma_0$, and the sides have surface charge $-\sigma_0$. Take the center of the cylinder as your origin, with the $z$-axis along the cylinder axis. Find the multipole moments of this distribution through $l=3$.

(Hint: You may find it useful to employ symmetry arguments, and to express $r^lP_l(\cos\theta)$ in Cartesian coordinates for calculations.)

**Question 9. Dipole moment of nuclei**

CALCULATION, DIPOLE MOMENT, REAL WORLD (C. Gwinn, UC Santa Barbara)

(a) Consider a charge distribution that contains only one sign of charge, and has a limited volume, such as an atomic nucleus. Let the charge density be $\rho(r)$ and suppose that the distribution is contained within a radius $a$ from the origin: $\rho = 0$ for $r > a$. Eq. 3.98 in Griffiths gives the dipole moment of the distribution, $p_{old}$.

You can translate the coordinates by the transformation $r_{new} = r_{old} + r_0$ where $r_0$ is a constant vector. Find the dipole moment of your distribution $p_{new}$ in the translated coordinates. Show that an appropriate choice of $r_0$ will yield zero
dipole moment $p_{\text{new}}$, if the charge density is all of one sign. For this reason, nuclei (and galaxies, which contain only positive gravitational charge) are sometimes said to have zero dipole moments; although strictly speaking this is true only for one choice of coordinates. (Thus, some nuclei have “dipole resonances” in which the proton “fluid” sloshes back and forth relative to the neutron “fluid”.)

b) Show that the dipole moment is independent of origin, if the total charge is zero. You may again assume that all charge is confined within a radius $a$.

One of the mysteries of particle phenomenology is the fact that the neutron (which has zero total charge, but contains quarks of both signs) has zero detectable dipole moment.

c) Show that any charge distribution within radius $a$ can be given zero dipole moment, by an appropriate choice of origin, so long as the total charge is nonzero. (Hint: Try superposing distributions from parts a and b). Note that the origin might be outside of radius $a$, in some cases.

**Question 10. Real dipole**

Griffiths 4.1 (p. 163) Only, instead of just using/assuming the Bohr radius (like Griffiths suggests) please estimate the radius of hydrogen, wherever you may need it, by using the experimental atomic polarizability of hydrogen given in Table 4.1 (p. 161) of Griffiths. (Griffiths' Example 4.1 should tell you how this would give you a quick estimate of the hydrogen atom radius. How does it compare with the Bohr radius?)

**Assigned in SP08 (average score: 8.43)**

Instructor notes: Students were not clear on where/how to start. The idea of using Griffiths' model problem (Ex 4.1) was not jumping out at them, I wonder if they actually read it? Seemed like maybe they had not. S. and R. were struggling just to visualize what the problem was asking in the first place. C. wanted a calculator, and when I told her this was an estimation problem, she seemed suddenly pleased, remembering e.g. that $\frac{1}{4}\pi \varepsilon_0 = k = 9\times10^9$, and so on...

**Question 11. Deep questions about multipole**

DEEPER QUESTIONS (From U WASHINGTON)

1.) Why should we care about approximate solutions for the scalar potential $V(r)$ and/or electric field $E(r)$? What are their benefits, and what are their uses?

2.) What are electric multipole moments of an electric charge distribution?

3.) When is it appropriate to use such approximate solutions the scalar potential $V(r)$ and/or electric field $E(r)$?

**Question 12. Non-linear dielectric**

We often assume that the induced dipole moment of an atom is proportional to the external field (at least for small fields). This is not a fundamental law, and it is easy to construct counter-examples, in theory. Suppose the charge density of the electron cloud is proportional to $r$ out to a maximum radius $R$. (Not a crazy assumption: in QM, the charge density is zero at the origin for non-s states.) In this case, the induced dipole moment $p$ is proportional to what power of the E, in the weak field limit?

**Assigned in FA08**
**Question 13. Boundary conditions**

In last week's problem 3 (Homework Set 6, problem 3), you had to solve for the voltage inside a square pipe with the voltage fixed on three sides of the pipe and the strange-looking condition $\frac{\partial V}{\partial x} = 0$ on the 4th side. What physical situation would produce such boundary conditions: that is, how would an experimentalist arrange things so as to guarantee that $\frac{\partial V}{\partial x} = 0$ on the 4th side? Hint: compare this problem with Griffiths’ example 3.4 on pages 132-134.

*Assigned in FA08*