Transformed E&M I homework

Vector Field (potential)  
(Griffiths Chapter 1)

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Vector identities (Chapter 1)

**Question 1. Cute checks of fundamental theorems**

**DIVERGENCE THEOREM (Griffiths 1.59)**

*Is there meaning to this math?*

Here are two cute checks of the fundamental theorems:

(a) Combine Corollary 2 to the gradient theorem with Stokes’ theorem (\( \mathbf{v} = \nabla T \), in this case). Show that the result is consistent with what you already knew about second derivatives.

(b) Combine Corollary 2 to Stokes’ theorem with the divergence theorem. Show that this is consistent with what you already knew.

**Question 2. Show various identities using vector identities**

**DIVERGENCE THEOREM (Griffiths 1.60)**

*Is there meaning to this math?*

Although the gradient, divergence and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) \[ \int (\nabla T) d\tau = \oint_s T d\mathbf{a} \]  [Hint: Let \( \mathbf{v} = eT \) where \( e \) is a constant, in the divergence theorem; use the product rules]

(b) \[ \int (\nabla \times \mathbf{v}) d\tau = -\oint_s \mathbf{v} \times d\mathbf{a} \]  [Hint: Replace \( \mathbf{v} \) by \( (\mathbf{v} \times e) \) in the divergence theorem]

(c) \[ \int [\nabla^2 U + (\nabla T) \cdot (\nabla U)] d\tau = \oint_s (\nabla U) \cdot d\mathbf{a} \]  [Hint: Let \( \mathbf{v} = TVU \) in the divergence theorem]

(d) \[ \int (\nabla^2 U - U \nabla^2 T) d\tau = \oint_s (\nabla U - U \nabla T) \cdot d\mathbf{a} \]  [Comment: This is known as Green’s theorem; it follows from (c) which is sometimes called Green’s identity]

(e) \[ \int_s \nabla T \times d\mathbf{a} = -\oint_p T d\mathbf{l} \]  [Hint: let \( \mathbf{v} = eT \) in Stokes’ theorem]

**Question 3. Show vector identity**

**VECTOR CALCULUS IDENTITIES, PROOF (From Lorrain, Corson and Lorrain, Electromagnetic Fields and Waves, Problem 1.6)**

Show that
\[ \int_V \nabla f \, dV = \oint_A f \, dA \] where A is the area of the closed surface bounding the volume V. Hint: multiply each side by a constant vector and use calculus theorems.

Question 4. Check Stokes’ theorem with line integral

STOKES’ THEOREM (Downloaded from Reed)

Consider the vector field written in cylindrical coordinates:

\[ A = s^2 \hat{\phi} \]

Sketch this vector function. Check Stokes’ theorem by calculating the line integral

\[ \oint A \cdot dl \]

around a circle of radius 2 centered at the origin and oriented in the x-y plane. Compare this to the surface integral of \( \nabla \times A \) over the interior of the circle.

Question 5. Evaluate surface integrals and verify Stokes

LINE, SURFACE, STOKES; A little more practice on vector calculus: Consider the vector function \( F(r) = \hat{\varphi} \) where \( \hat{\varphi} \) is defined as conventional for spherical coordinates.

a. Calculate the line integral

\[ \int_C F \cdot dl \]

where \( C \) is a circle of radius \( \rho \) in the xy plane, centered at the origin, and where the integral is evaluated with dl oriented counterclockwise.

b. Calculate the surface integral

\[ \int_H (\nabla \times F) \cdot da \]

where the surface \( H \) is a hemisphere that is above and bounded by the curve \( C \) used for part a. The surface integral is calculated with da oriented outward.

c. Calculate the surface integral

\[ \int_D (\nabla \times F) \cdot da \]

where \( D \) is now the disk in the xy plane that is bounded by curve \( C \), and da is oriented upward.

d. Verify that Stokes’ theorem holds for both surfaces \( H \) and \( D \)