Electromagnetic Waves in Conductors

In our previous discussions of EM waves in matter, we've been assuming there are no free charges ($\rho_f$) or free currents ($\mathbf{J}_f$); the only charges present were the bound charges in the dielectric. When dealing with conductors, there are plenty of electrons that are free to move about when an external electric field is applied. There are still bound charges in a conductor (atoms in a conductor generally only contribute one or two conduction electrons each), but we'll find that (in a good conductor) the interaction of the fields with the free charges and currents will dominate over everything else.

In an ohmic material (one that obeys Ohm's Law), we know that

$$\mathbf{J} = \sigma \mathbf{E}$$

where $\sigma$ is the conductivity of the conducting material (not the surface charge density!), which we'll take to be uniform inside the conductor. For any free charge that might build up in the conductor due to these currents, we know from charge conservation:

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$$

We can combine this with Gauss' Law for electric fields in matter $\nabla \cdot (\varepsilon \mathbf{E}) = \rho_f$, to find how quickly the free charge distribution would dissipate. Remember that Maxwell's equations are always true, in matter or not; splitting things up into free and bound charges (or currents) can be convenient for solving for the fields, but we don't have to do things this way.

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot (\sigma \mathbf{E}) = -\frac{\sigma}{\varepsilon} (\nabla \cdot (\varepsilon \mathbf{E})) = -\frac{\sigma}{\varepsilon} \rho_f$$

$$\rightarrow \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon} \rho_f$$

The solution to this first-order differential equation is:

$$\rho_f(t) = \rho_f(0) \exp\left(-\frac{\sigma}{\varepsilon} t\right)$$

This tells us that any free charge that might build up in a conductor will dissipate (due to the mutual repulsion of the charges) with a time constant $\tau = \varepsilon/\sigma$. The "time constant" represents the amount of time for the initial density to reduce to $1/e$ ($\sim 37\%$) of its original value. For a perfect conductor ($\sigma \to \infty$), the time
constant goes to zero, meaning the charge density instantly dissipates. For a “good” conductor (like copper), where the conductivity is typically on the order of $10^8 \, (\Omega \cdot m)^{-1}$, the time constant is somewhere around $10^{-19}$ seconds, which is actually much smaller than the typical time between collisions of the electrons with the atoms making up the conductor, which is $\tau_c \sim 10^{-14}$. For frequencies higher than $1/\tau_c$, Ohm’s law starts to break down, so $\tau_c$ is the important time constant here.

For the rest of this discussion, let’s assume we’re working with good conductors, and with frequencies that are at or below the optical range ($< 10^{15}$ Hz, which is pushing the use of Ohm’s law, but not so much that our results are no good). In other words, we should have $\omega \ll \sigma/\varepsilon$, or $\varepsilon\omega/\sigma \ll 1$. It turns out this assumption will make our calculations much easier as we go along.

Going back to Maxwell’s equations in matter, if we substitute in $\rho_f = 0$ (meaning free charges dissipate very quickly) and $\mathbf{J}_f = \sigma \mathbf{E}$, we have:

\[
\begin{align*}
(i) \quad & \nabla \cdot \mathbf{E} = 0 \\
(ii) \quad & \nabla \cdot \mathbf{B} = 0 \\
(iii) \quad & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
(iv) \quad & \nabla \times \mathbf{B} = \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mu \sigma \mathbf{E}
\end{align*}
\]

Taking the curl of $(iii)$ and $(iv)$ leads us once again to wave equations for $\mathbf{E}$ and $\mathbf{B}$ that are slightly different from those in vacuum:

\[
\nabla^2 \tilde{\mathbf{E}} = \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu \sigma \frac{\partial \mathbf{E}}{\partial t} \quad \text{and} \quad \nabla^2 \tilde{\mathbf{B}} = \mu \varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu \sigma \frac{\partial \mathbf{B}}{\partial t}
\]

The solutions to these equations are still plane waves:

\[
\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 \exp \left[ i \left( \tilde{k} z - \omega t \right) \right] \quad \text{and} \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 \exp \left[ i \left( \tilde{k} z - \omega t \right) \right]
\]

but now the wave number ($\tilde{k}$) is complex [it has a real and an imaginary part; that’s what the tilde ("squiggle") above the quantities in those equations is indicating]. We’ve seen complex wave numbers before when discussing our model for the index of refraction, so maybe you can already anticipate what happens to the amplitude of an EM wave as it penetrates into the conductor!

If we put our solutions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ back into our wave equations, we find:

\[
-\tilde{k}^2 \cdot \tilde{\mathbf{E}}(z,t) = -\mu \varepsilon \omega^2 \cdot \tilde{\mathbf{E}}(z,t) - i \mu \sigma \omega \cdot \tilde{\mathbf{E}}(z,t)
\]

\[
\longrightarrow \quad \tilde{k}^2 = \mu \varepsilon \omega^2 + i \mu \sigma \omega
\]
We can put this into the form $\tilde{k} = k_R + ik_I$, if we notice that:

$$\tilde{k}^2 = (k_R + ik_I)^2 = (k_R^2 - k_I^2) + i(2k_Rk_I) = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

We equate the real and imaginary parts to find:

$$k_R^2 - k_I^2 = \mu\varepsilon\omega^2 \quad \& \quad k_Rk_I = \mu\sigma\omega/2$$

Two equations with two unknowns allows us to solve for $k_R$ and $k_I$. It gets a little messy, but it turns out the final solution becomes:

$$k_R = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left( \sqrt{1 + \left( \frac{\sigma}{\varepsilon\omega} \right)^2} + 1 \right)^{1/2} \quad \& \quad k_I = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left( \sqrt{1 + \left( \frac{\sigma}{\varepsilon\omega} \right)^2} - 1 \right)^{1/2}$$

You can check this by substituting these back into the original equations. There's not much that's obvious from inspecting these equations for $k_R$ and $k_I$, but let's make use of our assumptions about working with a good conductor, and with frequencies at or below the optical range. This means that $\varepsilon\omega/\sigma \ll 1$, or equivalently: $\sigma/\varepsilon\omega \gg 1$. Suddenly, the equations look much simpler, for example:

$$k_R = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left( \sqrt{1 + \left( \frac{\sigma}{\varepsilon\omega} \right)^2} + 1 \right)^{1/2} \rightarrow \omega \sqrt{\frac{\varepsilon\mu}{2}} \cdot \sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\sigma\omega}{2}}$$

The real and imaginary parts of $\tilde{k}$ become equal, so we have:

$$\tilde{k} = \sqrt{\frac{\mu\sigma\omega}{2}} + i\sqrt{\frac{\mu\sigma\omega}{2}} = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i) = \sqrt{\mu\sigma\omega} \cdot \exp(i\pi/4)$$

Our solutions for $\tilde{E}$ and $\tilde{B}$ inside the conductor are now:

$$\tilde{E}(z,t) = \tilde{E}_0 \exp(-k_Iz)\exp\left[ i\left(k_Rz - \omega t\right) \right]$$

$$\tilde{B}(z,t) = \tilde{B}_0 \exp(-k_Iz)\exp\left[ i\left(k_Rz - \omega t\right) \right]$$

Just like with the attenuation of EM waves in a dispersive medium, there is a “skin depth” for conductors, defined like before as the distance over which the amplitude decreases to $1/e$ of its original value:

$$\exp(-k_Id) = \frac{1}{e} \rightarrow -1 = -k_Id \rightarrow d = \frac{1}{k_I}$$
The wave travels through the conductor with wave number \( k_R \). Let’s put some “typical” numbers in for these quantities to get a feel for their magnitudes:

\[
\begin{align*}
\omega & \sim 10^{15} \quad \sigma \sim 10^8 \quad \varepsilon \sim \varepsilon_0 \sim 10^{-11} \quad \mu \sim \mu_0 \sim 10^{-6}
\end{align*}
\]

\[
\rightarrow \quad d = \frac{1}{k_I} \approx \sqrt{10^6 \cdot 10^{-8} \cdot 10^{-15}} \sim 10^{-8} \text{ m}
\]

\[
\rightarrow \quad k_R \sim \sqrt{10^{-6} \cdot 10^{+8} \cdot 10^{+15}} \sim 10^8 \text{ m}^{-1}
\]

We’re justified in using \( \varepsilon \sim \varepsilon_0 \) because the dielectric constant for metals is not very high (meaning they’re good conductors, but not good dielectrics); we’re also assuming that our conductor has no magnetic properties (\( \mu \sim \mu_0 \)).

Notice that because \( k_R \) and \( k_I \) are of the same order of magnitude for a good conductor, the skin depth is about the same as a single wavelength (this result is actually independent of frequency). This means that metals of any significant thickness are opaque at optical frequencies (we already knew that!); optical wavelengths penetrate into a conductor no farther than a distance of \( \sim 100 \) atoms. But wavelengths for typical television and radio frequencies (\( \sim 1–100 \) meters) will pass through a thin sheet of metal relatively unaffected.

What about the relationship between \( \vec{E} \) and \( \vec{B} \) inside the conductor? From Faraday’s Law, we have:

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \rightarrow \quad i\hat{k} \times \vec{E} = i\omega\vec{B}
\]

\[
\rightarrow \quad \vec{B} = \frac{i}{\omega} \hat{k} \times \vec{E}
\]

We found before that \( \hat{k} = \sqrt{\mu \sigma \omega \exp(+i\pi/4)} \), and so:

\[
\vec{B} = \frac{i}{\omega} \hat{k} \times \vec{E} = \sqrt{\frac{\mu \sigma}{\omega}} \exp(+i\pi/4) \left( \hat{k} \times \vec{E} \right)
\]

where \( \hat{k} \) is just the unit vector pointing in the direction of \( k \) (it doesn’t matter if \( k \) is a complex number, it’s still pointing in a single direction like any other vector). We’re used to seeing \( E = cB \) for EM waves in vacuum, so let’s compare our result for conductors with the case for vacuum:

\[
B = \frac{\mu \sigma}{\omega} \cdot \varepsilon_0 \frac{1}{c} \sqrt{\frac{1}{\varepsilon_0 \mu_0 \cdot \omega}} = \frac{E}{c} \sqrt{\frac{\sigma}{\varepsilon_0 \omega}}
\]

\end{document}
Using the same “typical” values from before:

\[
\sqrt{\frac{\sigma}{\varepsilon_0 \omega}} \sim \sqrt{\frac{10^8}{10^{-11} \cdot 10^{15}}} = 100
\]

The magnetic field for an EM wave in a good conductor is about 100 times stronger for optical frequencies (and even much more, for lower frequencies) than for vacuum! This tells us that, since EM waves in vacuum have an energy that’s equal parts electric and magnetic, the energy for an EM wave in a conductor is almost entirely magnetic. This should make sense since the electrons being driven back and forth by the electric field are creating a free current in the conductor. Moreover, this factor of 100 came from looking at \( \omega/k \), which tells us that the wave speed for optical frequencies inside a good conductor is 100 times less than in vacuum, and even slower for lower frequencies!

We notice also that there’s a phase shift in the magnetic field relative to the electric field of \(+ \pi/4\). The B-field lags the E-field by 1/8 of a wavelength. [See Griffiths, Fig. 9.18, p. 396] This turns out to have some pretty important consequences.

Let’s think about the Lorentz force \( \mathbf{F} = q(\mathbf{E} + \mathbf{v}_q \times \mathbf{B}) \) on the electrons from an EM wave inside a good conductor (we’ll take the simplest case of normal incidence at the surface). We’ll write the electric and magnetic fields as:

\[
\mathbf{E} = E_0 \exp(-i\omega t) \hat{x}
\]

\[
\mathbf{B} = \frac{E_0}{\nu} \exp\left(-i\omega t + \frac{\pi}{4}\right) \hat{y}
\]

Remember, in this equation \( \nu \) is the speed of the traveling EM wave, and not the speed of the charges in the conductor (\( \nu_c \)). The electric field drives the charges at the oscillation frequency:

\[
q\mathbf{E} = m\ddot{x} \rightarrow x(t) = x_0 \exp(-i\omega t) \hat{x}
\]

\[
qE_0 = -m\omega^2 x_0 \rightarrow x_0 = \frac{qE_0}{m\omega^2} \exp(i\pi)
\]

The phase factor is due to the minus sign \( \exp(i\pi) = -1 \).
The velocity of the charges is:

\[ \vec{v}_q = \hat{x} \quad \rightarrow \quad v_{q0} = -i\omega x_0 = \frac{qE_0}{m}\omega \exp(i\pi/2) \]

since \( \exp(-i\pi/2) = -i \).

Putting this result into the Lorentz force law, where we’re using only the real parts of the fields and the velocities:

\[
\vec{F} = qE_0 \cos(\omega t)[\hat{x}] + \frac{q^2E_0}{m\omega} \cos(\omega t + \pi/2) \cdot \frac{E_0}{v} \cos(\omega t + \pi/4)[\hat{x} \times \hat{y}]
\]

\[
= qE_0 \cos(\omega t)[\hat{x}] + \frac{q^2E_0^2}{m\omega v} \cos(\omega t + \pi/2)\cos(\omega t + \pi/4)[\hat{z}]
\]

The time-average of the electric force is zero, because the cosine averages out to zero:

\[
\langle F_E \rangle = qE_0 \langle \cos(\omega t) \rangle \hat{x} = 0
\]

What about the magnetic force, which is the product of two cosine functions that are out phase by \( \pi/4 \)? Let’s rewrite the cosines without the phase shifts:

\[
\cos(\omega t + \pi/2) = -\sin(\omega t)
\]

\[
\cos(\omega t + \pi/4) = \frac{1}{\sqrt{2}} [\cos(\omega t) - \sin(\omega t)]
\]

The factors multiplying the cosines in the magnetic force are all positive, and the value of the charge is squared, so it doesn’t matter if we’re talking about positive or negative charges. Therefore:

\[
\langle F_B \rangle = \frac{|F_B|}{\sqrt{2}} \langle -\sin(\omega t)\cos(\omega t) + \sin^2(\omega t) \rangle \hat{z}
\]

We’ve used \( |F_B| \) as shorthand for the amplitude of the time-varying magnetic force. The time-average of \( \sin(\omega t)\cos(\omega t) \) is also zero, but \( \langle \sin^2(\omega t) \rangle = 1/2 \), so:

\[
\langle F_B \rangle = \frac{|F_B|}{2\sqrt{2}} \hat{z}
\]

The electric field drives the electrons up and down, but the magnetic field doesn’t just drive them back and forth, over time it drives them forward in the +z-direction. This provides a (classical) explanation for the origin of radiation pressure!
Notice that if the E- and B-fields had been in phase with each other, then the time-averaged magnetic force would have been:

$$\langle F_B \rangle = \frac{|F_B|}{\sqrt{2}} \langle -\sin(\omega t)\cos(\omega t) \rangle = 0$$

because the field and the velocity of the charges are out of phase by $\pi/2$.

Finally, let’s discuss the reflection of EM waves at the surface of a conductor. We already know from our previous talk about boundary conditions that the angle of reflection must equal the angle of incidence (no matter the material), so for simplicity we’re only going to talk about normal incidence. Our boundary conditions are:

$$\begin{align*}
(i) & \quad \varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = \sigma_f \\
(ii) & \quad B_1^\perp - B_2^\perp = 0 \\
(iii) & \quad E_1^\parallel - E_2^\parallel = 0 \\
(iv) & \quad \frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel = K_f \times \hat{n}
\end{align*}$$

where $\sigma_f$ is the free surface charge (not to be confused with the conductivity!), $K_f$ is the free surface current, and $\hat{n}$ is a unit vector pointing perpendicular to the surface. At normal incidence, (i) and (ii) are irrelevant since there are no perpendicular components of the fields. For ohmic materials, $\mathbf{J} = \sigma \mathbf{E}$ says that in order to get a surface current that looks like a “delta function”, you’d need to have an infinite E-field there (which is unphysical); there is a free current inside the conductor, but none right on the boundary. In other words, $K_f = 0$ in this situation.

Notice that now the boundary conditions are exactly like when we were considering reflection and transmission at the surface of a dielectric. This means our formulas for the reflected and transmitted waves are going to be the same, but we’ll have to fix them up slightly to take into account that the index of refraction is going to be complex (having real and imaginary parts). For an EM wave incident from vacuum, these equations therefore become:

$$\tilde{E}_R = \frac{1 - \hat{n}}{1 + \hat{n}} \tilde{E}_I \quad \& \quad \tilde{E}_T = \frac{2}{1 + \hat{n}} \tilde{E}_I$$

For a perfect conductor ($\sigma \to \infty$), the complex index of refraction becomes infinite (remember, our formula for $\tilde{k}$ was proportional to $\sqrt{\sigma}$), and so:

$$\tilde{E}_R = -\tilde{E}_I \quad \& \quad \tilde{E}_T = 0$$

For a perfect conductor, the free electrons can move in perfect response to the incident wave, and the transmitted wave goes to zero. The wave is entirely reflected with a phase shift of $180^\circ$. 