A Simple Model for the Index of Refraction

Assume we can model a linear dielectric as being made from atoms that behave like a charge $q$ attached by a spring to an oppositely charged nucleus that’s fixed in place (a simple harmonic oscillator). This turns out to be a fairly good assumption, because quantum mechanics tells us that, as far as problems involving light are concerned, the electrons in an atom behave as though they are harmonic oscillators. With no externally applied field, let’s say the charge is located at its equilibrium position (no net force on the charge). The binding force ($\vec{F}_B$) is given by Hooke’s Law, which says the restoring force is proportional to the amount of stretch in the spring:

$$\vec{F}_B = -k_s \vec{x} = -m\omega_0^2 \vec{x}$$

$k_s$ is the spring constant (not to be confused with the wave number!), $m$ is the mass of the charge, and we’ve defined the “natural frequency” $\omega_0 = \sqrt{k_s/m}$ as usual for a harmonic oscillator.

If we apply a uniform electric field $\vec{E}_0$, the spring is stretched until the binding force balances the applied electric force ($\vec{F}_E = q\vec{E}_0$). If the spring is stretched by a maximum distance $\vec{d}$, then

$$\vec{F}_E + \vec{F}_B = 0 \quad \rightarrow \quad q\vec{E}_0 = m\omega_0^2 \vec{d}$$

$$\vec{d} = \frac{q}{m\omega_0^2} \vec{E}_0 \quad \text{which is the same as} \quad \vec{d} = \frac{q}{k_s} \vec{E}_0$$

For two equal and opposite charges ($q$ & $-q$) separated by a distance $\vec{d}$, the dipole moment is defined as $\vec{p} = q\vec{d}$, and so for this atom:

$$\vec{p} = \frac{q^2}{m\omega_0^2} \vec{E}_0$$

If $N$ is the number of atoms per unit volume in the dielectric, then the polarization $\vec{P}$, which is the total dipole moment per unit volume in the dielectric:

$$\vec{P} = \frac{\text{total dipole moment}}{\text{volume}} = N \cdot \vec{p} = \frac{Nq^2}{m\omega_0^2} \vec{E}_0$$

Notice that the polarization vector points in the same direction as the applied electric field (all the terms out front are positive), but the electric field due to the
charge separation in the atom points in the direction opposite of \( \mathbf{P} \), which tends to reduce (cancel) the applied electric field. You may recall from the homework that if we think about a polarized atom as two overlapping spheres (one positively charged, the other negative) displaced from each other by a small distance, then the electric field in the region where the sphere’s overlap (\( \mathbf{E}_p \)) is uniform, and points opposite the applied field.

In a linear dielectric, the applied external field induces a polarization (separation of charge) in the dielectric material that tends to reduce the total electric field in the region occupied by the material. We’re assuming here that each atom is only responding to the externally applied field, and not to the dipole fields of the other atoms (which is approximately true when they’re not too closely spaced together).

What factors will increase the polarization in the material?

- \( \mathbf{P} \sim \mathbf{E}_{\text{ext}} \) Apply a stronger external electric field.
- \( \mathbf{P} \sim N \) Use a denser material (increase \( N \)).
- \( \mathbf{P} \sim \frac{1}{m\omega_0^2} = \frac{1}{k_s} \) Use atoms where the charges are less tightly bound (smaller spring constant, smaller binding force). Let’s assume we can’t change the mass of our charges.

That’s for the case of a constant, uniform electric field. What if we apply a time-varying electric field, like an EM plane wave?

\[
\mathbf{E}_{\text{ext}} = \mathbf{E}_0 \cos(\omega t) \quad \rightarrow \quad \mathbf{E}_0 \exp(-i\omega t)
\]
Why the minus sign in \((-i\omega t)\)? Remember that we usually write for a wave traveling in the positive \(z\) direction: \((kz - \omega t)\) [we can let \(z = 0\) in this case, and not worry about any phase shifts].

The driving force \((\vec{F}_D)\) on the charge due to the applied electric field is:

\[
\vec{F}_D = q\vec{E}_0 \exp(-i\omega t)
\]

So, the total force on the charge is (and using Newton’s second law):

\[
\vec{F}_{TOTAL} = \vec{F}_D + \vec{F}_B = m\ddot{\vec{a}} \quad \rightarrow \quad m\frac{d^2x}{dt^2} = qE_0 \exp(-i\omega t) - m\omega_0^2x
\]

\[
\rightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2x = \frac{qE_0}{m}\exp(-i\omega t)
\]

We assume a solution for \(x(t)\) that oscillates at the driving frequency \(\omega\):

\[
x(t) = x_0 \exp(-i\omega t)
\]

Substituting this into our differential equation, we find:

\[
-\omega^2 x_0 \exp(-i\omega t) + \omega_0^2 x_0 \exp(-i\omega t) = \frac{qE_0}{m}\exp(-i\omega t)
\]

All the exponentials cancel out, and we can solve for \(x_0\):

\[
(\omega_0^2 - \omega^2)x_0 = \frac{qE_0}{m} \quad \rightarrow \quad x_0 = \frac{qE_0}{m(\omega_0^2 - \omega^2)}
\]

\(x_0\) is the amplitude (maximum displacement) of the oscillating charge. Since the dipole moment is defined as the magnitude of the charge times the distance of separation:

\[
\vec{p}(t) = q \ddot{x}(t) = \frac{q^2}{m(\omega_0^2 - \omega^2)}\vec{E}(t)
\]

This is just for one “atom”, so we again multiply this by the \(N\) atoms per unit volume to get the polarization (dipole moment per unit volume). For now, let’s say that we’re working with driving frequencies that are small compared to the natural frequency of the atom, which tends to be in the UV range (higher than optical frequencies). We can then assume that \(\omega \ll \omega_0\), so that:
\[ \mathbf{P}(t) = N \cdot \mathbf{p}(t) = \frac{Nq^2}{m\omega^2} \mathbf{E}(t) \]

The relationship between the polarization and the dielectric constant (\(\varepsilon_r\)) is:

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad (\chi_e = \text{electric susceptibility}) \]

\[ \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (\varepsilon = \text{permittivity of the material}) \]

\[ \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} \]

\[ \varepsilon_r = \"\text{dielectric constant}\" \equiv 1 + \chi_e \]

\[ \rightarrow \quad \varepsilon_r = 1 + \frac{1}{\varepsilon_0} \frac{\mathbf{P}}{\mathbf{E}} = 1 + \frac{Nq^2}{\varepsilon_0 m\omega^2} \]

For dilute gases (at standard pressure and temperature), the dielectric constant is going to be a number that's very close to 1. [See Griffiths, Table 4.2, p. 180]

<table>
<thead>
<tr>
<th>Material</th>
<th>Dielectric Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>1.000065</td>
</tr>
<tr>
<td>Air (dry)</td>
<td>1.00054</td>
</tr>
<tr>
<td>Water vapor</td>
<td>1.00587</td>
</tr>
</tbody>
</table>

For dilute gases, \(\chi_e\) is therefore a number that's very small compared to 1. The index of refraction is given by:

\[ n = \frac{c}{v} = \sqrt{\varepsilon_0 \mu_0} = \left( \frac{1}{\varepsilon_0 \mu_0} \right) \left( \sqrt{\frac{\varepsilon}{\mu}} \right) = \sqrt{\frac{\varepsilon}{\varepsilon_0 \mu_0}} \]

and we can further assume that the dielectric material doesn't have any magnetic properties (\(\mu = \mu_0\)):

\[ n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r} \]
\[ n = \sqrt{1 + \chi_e} = (1 + \chi_e)^{1/2} \approx 1 + \frac{1}{2} \chi_e \]

where we’ve expanded the square root in a binomial series and only keep the first-order term, since higher powers of \( \chi_e \) will be even smaller.

\[ n = 1 + \frac{Nq^2}{2\varepsilon_0 m\omega_0^2} \]

The index of refraction is larger for more dense materials (larger \( N \)), or when the charges are less tightly bound (small \( k_s = \) small \( \omega_0 \)), which leads to a greater charge separation. Increasing either of these factors leads to more polarization.

The index of refraction is closer to 1 (more like a vacuum) for less dense materials, or when the atoms are less polarizable. In this sense, the index of refraction is a measure of how much a material is “not like a vacuum”.

Remember the equations we derived for the reflected \( (E_R) \) and transmitted \( (E_T) \) waves, when light is traveling from vacuum \( (n = 1) \) to a material with a higher index of refraction:

\[ E_T = \frac{2}{1 + n} E_I \quad \& \quad E_R = \frac{1 - n}{1 + n} E_I \]

For the transmitted wave, whenever \( n > 1 \), we have \( E_T < E_I \). This partial cancellation of the incident electric field is due to the polarization in the dielectric. Lots of polarization (large \( n \)) leads to lots of cancellation of the incident electric field, and therefore a smaller transmitted electric field.

But the index of refraction is also related to the wave speed in the dielectric \( (v = c/n) \), so \( n > 1 \) means light travels slower in that medium. Putting this all together, light travels slower in a medium with higher density (large \( N \)) or where the atoms are more easily polarized (small \( k_s = \) small \( \omega_0 \)). This should make sense if we recall that we see a similar kind of relationship for the speed of a wave traveling on a string:

\[ v_{\text{STRING}} = \sqrt{\frac{\text{Tension}}{\text{mass density}}} \quad \leftrightarrow \quad \omega_0 = \sqrt{\frac{k_s}{m}} \]

The more tension (restoring force) in the string, the faster the wave propagation. Analogously, the stiffer the “atomic spring” (large \( k_s = \) large \( \omega_0 \)), the closer the EM wave speed is to \( c \) (\( n \) closer to 1).
What about the reflected wave?

\[ E_R = \frac{1 - n}{1 + n} E_I \quad \rightarrow \quad E_R < 0 \quad \text{when} \quad n > 1 \]

A minus sign is equivalent to a phase shift of 180°, since \( \exp(i\pi) = -1 \).

From the usual boundary condition on the parallel components of the electric field, it's required that the total electric field parallel to the interface be continuous across the boundary:

\[ E_I + E_R = E_T \]

The reflected wave is always 180° out of phase with the incident wave whenever the transmitted wave has a smaller amplitude than the incident wave.

This happens whenever light goes from a lower to a higher index of refraction, i.e., whenever light passes into a medium where it travels slower.

The smaller the amplitude of the transmitted wave, the greater the amplitude of the reflected wave.

The closer the material is to being like vacuum (\( n \) close to 1, meaning less density and/or less polarizability) the more the light behaves as though it were in vacuum (\( v \) closer to \( c \)).

Now, let's go back to where we made our assumptions about a driving frequency that was small compared to the natural frequency of the oscillator:

\[ n = 1 + \frac{Nq^2}{2\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2} \]

Notice there's an explicit dependence on frequency in the index of refraction (this is known as “dispersion”, for reasons that will be clear later). This says \( n \) will be different for different incident frequencies (which is behind what we see when white light is passed through a prism). This frequency dependence will be most prominent near the natural frequency (resonance frequency) of the charges in the atoms making up the dielectric. But there's a problem in our formula, because it says that the index of refraction will become \textit{infinite} whenever \( \omega = \omega_0 \).
If our formula were correct, the index of refraction as a function of frequency would look like:

![Graph 1](image1)

What we actually see if we do this experiment is more like:

![Graph 2](image2)

So we need to fix up our model, because it doesn’t quite correspond yet to what we actually observe. We can do this if we recognize that there’s some kind of damping force on the oscillating charges. This is due to the fact that accelerated charges radiate – the oscillating charges radiate away energy in the form of electromagnetic waves. Because the total energy of the system is conserved, there must be some kind of “radiation force” that slows the oscillation of the charges as they radiate energy. We’ll worry later about deriving the equation for this radiation force, but it turns out that it’s proportional to the time-derivative of the acceleration of the charge:
\[ F_{\text{RAD}} = \frac{\mu_0 q^2}{6\pi c} \frac{da}{dt} \]

But if we’re assuming an oscillatory solution for \( x(t) \):

\[
x(t) = x_0 \exp(-i\omega t) \quad \rightarrow \quad \frac{da}{dt} = \frac{d^3x}{dt^3} = \frac{d}{dt}(-\omega^2 x) = -\omega^2 \frac{dx}{dt}
\]

we might just as well assume the radiation force is proportional to the velocity of the charge:

\[ F_{\text{DAMPING}} = -m\gamma \frac{dx}{dt} \quad [\gamma = \text{“damping constant”}] \]

Our differential equation for this damped harmonic oscillator is now:

\[
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{qE_0}{m} \exp(-i\omega t)
\]

We solve for \( x(t) \) in the same way as before and find:

\[
\ddot{x}_0 = \frac{qE_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}
\]

We now have for the index of refraction:

\[
\tilde{n} = 1 + \frac{Nq^2}{2m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}
\]

Right away, we can see that the index of refraction will be finite for any \( \omega \). [good!]

Notice that \( n \) is now a complex number (it has a real and an imaginary part). Let’s rationalize the denominator (multiply top and bottom by the complex conjugate).

\[
\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \cdot \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{\omega_0^2 - \omega^2 + i\gamma\omega} = \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}
\]

We’ll focus first on just the real part of \( \tilde{n} = n_R + in_I \)

\[
n_R \approx 1 + \frac{Nq^2}{2m\varepsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}
\]

This has just the kind of behavior we observe. When the incident frequency is increasing towards the “resonance frequency” \( \omega_0 \), the real part of the index of
refraction increases suddenly, but doesn’t become infinitely large. Just past the resonance frequency, it dips down dramatically, and the real part of the index of refraction can even become less than 1 (more about this later).

The index of refraction in a dispersive material will depend on the frequency of the wave, and so different frequencies will travel at different speeds. A wave packet made from many different frequencies will spread out, some frequencies traveling faster through the medium, some frequencies traveling slower [hence the name “dispersion”]. The two plots below show the propagation of a wave packet through a dispersive medium with a frequency right at the resonant frequency; the plot on the left shows the wave packet at $t = 0$, and the one on the right shows the wave packet a short time later ($t > 0$). Notice how the spatial extent of the packet has spread out. The amplitude is also decreased – we’ll address this in a moment.

![Wave Packet Propagation](image)

The sudden drop in $n$ below 1 at frequencies just above the resonant frequency is known as “anomalous dispersion”. It’s called “anomalous” because it’s different than what we expect, but also because it was discovered after the so-called “normal dispersion” (as with a prism).

What about the imaginary part of the index of refraction? Does it have a physical meaning as well? The answer is: Yes!

$$ n_I = \frac{N q^2}{2 m \varepsilon_0} \left( \frac{\gamma \omega}{\omega_0^2 - \omega^2} + \frac{\gamma^2 \omega^2}{\omega_0^2 - \omega^2} \right) $$

Recall that $n = \frac{c}{v} = \frac{c k}{\omega}$, where $k$ is the wave number (not the spring constant!).

$$ \tilde{k} = \frac{\omega}{c} \tilde{n} = \frac{\omega}{c} (n_R + i n_I) $$
So, the exponential term in our expression for the electric field in the dielectric looks like:

\[
\exp\left[i(kz - \omega t)\right] = \exp\left[i\frac{\omega n_R}{c} z - i\omega t\right]
\]

\[
\rightarrow \quad \exp\left(-\frac{\omega n_I}{c} z\right) \cdot \exp\left[i\frac{\omega n_R}{c} z - i\omega t\right]
\]

The second term corresponds to a wave traveling to the right through the dispersive medium with a wave number \( k_R = \omega n_k / c \).

The first term is exponentially decaying with increasing \( z \), meaning the amplitude of the transmitted wave decreases exponentially as it penetrates into the dielectric:

\[
\exp(-k_I d) = \frac{1}{e} \quad \rightarrow \quad -1 = -k_I d \quad \rightarrow \quad d = \frac{c}{\omega n_I}
\]

We can define a characteristic length for how far the wave penetrates into the material, by defining it as the distance over which the amplitude decreases to 1/e of its original value – this distance is known as the “skin depth” of the material.
Putting this all together, we see that the complex index of refraction contains two pieces of information: the real part tells us the speed of propagation for a traveling wave inside the dielectric; the imaginary part tells us how the amplitude of the wave gets attenuated as it penetrates into the dielectric. Because the oscillators are radiating energy, the amplitude of the wave is diminished (energy is being absorbed by the material), where the greatest absorption occurs at the resonance frequency. Just below the resonance frequency, the real part of the index of refraction dramatically increases; just above this frequency it then drops below 1 (anomalous dispersion).

Before we address what’s going on when \( n < 1 \), let’s make some final generalizations to our model for the index of refraction. The model can be made more general by assuming there’s more than one electron attached to a nucleus, and that each electron in an atom can have a different binding energy; we could also assume that each electron has its own damping constant. Call the natural frequency for the \( j \)-th electron \( \omega_j \), and similarly for its damping constant \( \gamma_j \). The total index of refraction will come from summing up all the contributions from each electron:

\[
n = 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{\omega_j^2 - \omega^2 + i\gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}
\]

This means we’ll see a series of peaks (and dips) in the index of refraction, each occurring at the different resonant frequencies of the dielectric. The index of refraction is a sum over many different terms, each of which doesn’t change much when \( \omega \) is far from the resonant frequency of that term. This means there’s a fairly constant “background” that can sometimes lead to the index of refraction never dipping below 1, but it’s going to depend on the actual contributions from all the different terms, which is specific to the material. Ultimately, we need to use quantum mechanics to determine what the different resonant frequencies should be.
for a given material. They can certainly be determined experimentally, just by varying the frequency of light incident on the material and looking for absorption resonances.

Let's summarize the assumptions we made while coming up with this model:

1. The charges in an atom act like damped harmonic oscillators in response to an applied electric field that is not too strong.

2. The dielectric has no magnetic properties ($\mu = \mu_0$).

3. The electric susceptibility ($\chi_e$) is small compared to 1 (meaning we're working with a dilute gas, or atoms that are not easily polarized, or both).

We might think we get into trouble with the last one, since there's a resonance effect that can make the susceptibility much bigger, but if we need to we can always keep more terms in our expansion of $\sqrt{1 + \chi_e}$, which would just make our formula a bit more complicated. Overall, this is a decent model for the index of refraction under these specific conditions, and it also gives us some insight into the behavior of materials that don't meet these assumptions. For such cases, the final formula for the index of refraction may look somewhat different, but we should still expect to see the same kind of qualitative behavior we find from examining this simplified model.