EM fields & Conservation of Energy

What does it mean to say something is "locally conserved"? It means that, if the amount of that something in one region changed, it had to be because some of that something moved to an adjacent region. The stuff conserved can't just disappear in one place and instantly reappear in another far-away place; it has to move.

Charge is locally conserved:

If charge inside \( V \) decreases \( \Rightarrow \) charge had to cross surface \( S(V) \) from inside to outside.

\[
- \frac{d}{dt} \left( \int_V \rho \, dV \right) = \oint_{S(V)} \mathbf{j} \cdot d\mathbf{a}
\]

\( Q_{\text{enc}} \) units: \( \frac{\text{C}}{\text{m}^2} = \text{A} = \frac{\text{Q}}{\text{t}} \)

\[
\Leftrightarrow -\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j}
\]

\[
\mathbf{j} = \mathbf{v} \times \mathbf{B} \quad \Rightarrow \quad \mathbf{v} = \frac{\text{charge}}{\text{area} \cdot \text{time}}
\]

In general, if "stuff" is conserved (locally)

\[
- \frac{\partial}{\partial t} \text{(stuff)} = \nabla \cdot \text{(stuff current)}
\]

We know energy is conserved. Consider a collection of charges and fields, where all the charges stay within volume \( V \):
outside V, there can be fields (but no charges)

inside V are all the charges + fields

we know fields contain energy and we know fields can carry volume V energy (you’ve felt warm sunlight on your face).

Define vector \( \vec{S} = \vec{E} \times \vec{B} \) EM energy flow density

\[ |\vec{S}| = \text{energy} \quad \frac{\text{area} \cdot \text{time}}{} \]

\[ \frac{\partial}{\partial t} \frac{\vec{S} \cdot \Delta \vec{A}}{\Delta A} = \left( \frac{\text{EM field energy}}{\text{time}} \right) \text{ passing thru } \Delta \vec{A} \]

We seek \( \vec{S} \) such that

\[ -\frac{d}{dt} \left( \text{total energy inside } V \right) = \oint \vec{S} \cdot d\vec{A} \]

Now, \( (E_{\text{tot inside } V}) = \int u \, d\tau \), \( u = \text{energy/vol} \)

\[ u = u_{\text{em}} + u_{\text{mech}}, \text{ where} \]

\[ u_{\text{em}} = \frac{\text{energy in fields}}{\text{vol}} = \frac{E_0}{2} E^2 + \frac{1}{2 \mu_0} B^2 \]

\[ u_{\text{mech}} = \text{ every other kind of energy/vol} \]

\[ = KE \text{ of particles} + PE_{\text{gravity}} + PE_{\text{electrostatic}} + \ldots \]

\( (PE_{\text{electric already included in } u_{\text{em}}}) \) No!
For simplicity, let's consider $u_{mech}$ to just be the particle KE/vol (in general, it could also include thermal energy, grav. PE, etc.)

So, we seek $\mathbf{S}$ that satisfies:

$$-rac{d}{dt} \left( \int_{V} \left( u_{em} + u_{mech} \right) d\tau \right) + \oint_{\partial V} \mathbf{S} \cdot d\mathbf{a} = 0$$

Note: $\mathbf{S}$ in differential form,

$$-\frac{\partial u_{em}}{\partial t} - \frac{\partial u_{mech}}{\partial t} = \nabla \cdot \mathbf{S} \quad (\star)$$

We can relate $\frac{\partial u_{mech}}{\partial t}$ to $\mathbf{E}$ and $\mathbf{j}$ like so:

Work done by fields on charges = energy gained by charges

$$dW = \mathbf{E} \cdot d\mathbf{l} = q \left( E + \mathbf{v} \times B \right) \cdot \mathbf{v} dt = q \cdot \mathbf{E} \cdot \mathbf{v} dt$$

Power = $\frac{dW}{dt} = q \mathbf{E} \cdot \mathbf{v} = \mathbf{E} \cdot \left( \rho \mathbf{v} \right) d\tau = \mathbf{E} \cdot \mathbf{j} d\tau$

So rate at which all particles in $V$ are gaining energy = total power in $V$:

$$\int_{V} \mathbf{E} \cdot \mathbf{j} d\tau = \frac{d}{dt} \left( \int_{V} u_{mech} d\tau \right)$$

$$\Rightarrow \mathbf{E} \cdot \mathbf{j} = \frac{\partial u_{mech}}{\partial t}$$
We're almost there. Seek $S'$ (EM energy flow density) that obeys eq'n (8) which is:

$$\nabla \cdot \vec{S}' = -\vec{E} \cdot \vec{J} - \frac{\partial}{\partial t} \left( \frac{\varepsilon_0 E^2}{2} + \frac{1}{2\mu_0} B^2 \right)$$  (8)

We'll need product rule (6) from Griffiths:

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B})$$  [PR61]

(We might guess $\vec{E} \times \vec{B}$ has something to do w/ $S'$, since $S'$ must involve fields and $S'$ is a vector.)

Faraday's Law $\Rightarrow \vec{B} \cdot (\nabla \times \vec{E}) = \vec{B} \cdot (-\frac{\partial \vec{B}}{\partial t})$

Maxwell-Amper' $\Rightarrow \vec{E} \cdot (\nabla \times \vec{B}) = \mu_0 \vec{E} \cdot \vec{J} + \mu_0 \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$

Now $\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial B^2}{\partial t}$

And $\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}$

[PR61] $\Rightarrow \nabla \cdot \left( \frac{1}{\mu_0} \vec{E} \times \vec{B} \right) = -\frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} - \vec{E} \cdot \vec{J} - \frac{\varepsilon_0 E^2}{2} \frac{\partial}{\partial t}

\text{or } \nabla \cdot \left( \frac{1}{\mu_0} \vec{E} \times \vec{B} \right) = -\vec{E} \cdot \vec{J} - \frac{1}{2} \left( \frac{\varepsilon_0 E^2}{2} + \frac{1}{2\mu_0} B^2 \right)

\text{Comparing w/ (8) we see}

$$S = \frac{i}{\mu_0} \vec{E} \times \vec{B}$$  = energy carried by fields

area \times time