Problem Set –Chapter 6 Solutions

1. Ch 6, Problem 6.1
A firm uses the inputs of fertilizer, labor, and hothouses to produce roses. Suppose that when the quantity of labor and hothouses is fixed, the relationship between the quantity of fertilizer and the number of roses produced is given by the following table:

<table>
<thead>
<tr>
<th>Tons of Fertilizer per Month</th>
<th>Number of Roses per Month</th>
<th>Tons of Fertilizer per Month</th>
<th>Number of Roses per Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>2500</td>
</tr>
<tr>
<td>1</td>
<td>500</td>
<td>6</td>
<td>2600</td>
</tr>
<tr>
<td>2</td>
<td>1000</td>
<td>7</td>
<td>2500</td>
</tr>
<tr>
<td>3</td>
<td>1700</td>
<td>8</td>
<td>2000</td>
</tr>
<tr>
<td>4</td>
<td>2200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. What is the average product of fertilizer when 4 tons are used?

\[ AP_F = \frac{Q}{F} = \frac{2200}{4} = 550. \]

b. What is the marginal product of the sixth ton of fertilizer?

\[ MP_F = \frac{\Delta Q}{\Delta F} = \frac{2600 - 2500}{6 - 5} = 100 \]

c. Does this total product function exhibit diminishing marginal returns? If so, over what quantities of fertilizer do they occur?

Diminishing marginal returns set in when \( MP_F \) for some unit is lower than \( MP_F \) for the previous unit. This occurs for \( F > 3 \).

d. Does this total product function exhibit diminishing total returns? If so, over what quantities of fertilizer do they occur?

Diminishing total returns set in at the point where total output begins to fall. This occurs for \( F > 6 \).

2. Ch 6, Problem 6.5

Are the following statements correct or incorrect?
a. If average product is increasing, marginal product must be less than average product.
Incorrect. When $MP > AP$ we know that $AP$ is increasing. When $MP < AP$ we know that $AP$ is decreasing.

b. If marginal product is negative, average product must be negative.

Incorrect. If $MP$ is negative, $MP < 0$. But $AP = Q / L$ can never be negative since total product $Q$ and the level of input $L$ can never be negative. Thus, $MP < 0 < AP$, which only implies that $AP$ is falling.

c. If average product is positive, total product must be rising.

Incorrect. Average product is always positive, so this tells us nothing about the change in total product. Whether or not total product is rising depends on whether or not marginal product is positive.

d. If total product is increasing, marginal product must also be increasing.

Incorrect. If total product is increasing we know that $MP > 0$. If diminishing marginal returns have set in, however, marginal product will be positive but decreasing, but that does not preclude $MP > 0$.

3. Ch 6, Problem 6.4

Suppose the production function for lava lamps is given by

$$Q = KL^2 - L^3,$$

where $Q$ is the number of lamps produced per year, $K$ is the machine-hours of capital, and $L$ is the man-hours of labor.

Suppose $K = 600$.

a. Draw a graph of the production function over the range $L = 0$ to $L = 500$, putting $L$ on the horizontal axis and $Q$ on the vertical axis. Over what range of $L$ does the production function exhibit increasing marginal returns? Diminishing marginal returns? Diminishing total returns?
To identify the region of increasing total returns and the region of diminishing total returns, we need to take the first derivative of the production function:

$$MP_L = \frac{\partial Q}{\partial L} = 1200L - 3L^2.$$  

When $MP_L > 0$, the production function is exhibiting increasing total returns; when $MP_L < 0$, the production function is exhibiting diminishing total returns. Therefore, the we have increasing total returns when $0 < L < 400$, and we have diminishing total returns when $400 < L < 500$.

To identify the region of increasing marginal returns and the region of diminishing marginal returns, we need to take the second derivative of the production function:

$$\frac{dMP_L}{dL} = \frac{\partial^2 Q}{\partial L^2} = 2400 - 6L.$$  

When $dMP_L/dL > 0$, the production function is exhibiting increasing marginal returns; when $dMP_L/dL < 0$, the production function is exhibiting diminishing marginal returns. Therefore, we have increasing marginal returns when $0 < L < 200$, and we have diminishing marginal returns when $200 < L < 500$.

b. Derive the equation for average product of labor and graph the average product of labor curve. At what level of labor does the average product curve reach its maximum?

The equation for average product of labor is given by
\[ AP_L = \frac{Q}{L} = \frac{600L^2}{L} - \frac{L^3}{L} = 600L - L^2. \]

Graphing this equation yields the average product of labor curve:

To find the level of labor at which the average product curve reaches its maximum, we need to take the derivative and set it equal to zero:

\[ \frac{\partial AP_L}{\partial L} = 600 - 2L \]
\[ 600 - 2L = 0 \]
\[ L = 300 \]

So, the average product curve reaches its maximum when \( L = 300 \).

c. Derive the equation for marginal product of labor. On the same graph you drew for part b, sketch the graph of the marginal product of labor curve. At what level of labor does the marginal product curve appear to reach its maximum? At what level does the marginal product equal zero?

We derived the equation for marginal product of labor in part a. We rewrite it here for convenience:

\[ MP_L = 1200L - 3L^2. \]

Graphing this equation yields the marginal product of labor curve. It is shown below in pink (the average product of labor curve is shown in yellow):
To find the level of labor at which the marginal product curve reaches its maximum, we need to take the derivative and set it equal to zero:

\[ \frac{\partial MP_L}{\partial L} = 1200 - 6L \]

\[ 1200 - 6L = 0 \]

\[ L = 200 \]

So, the marginal product curve reaches its maximum when \( L = 200 \).

The marginal product equals zero when \( L = 400 \).

d. Relate your answer to part c to your answer to part a.

In part c, we saw that marginal product is positive over the range \( 0 < L < 400 \). This means that increases in labor lead to increases in output over this range. So it follows that this range coincides with the region of increasing total returns identified in part a.

In part c, we saw that marginal product is equal to zero when \( L = 400 \) and is then negative over the range \( 400 < L < 500 \). This means that increases in labor lead to \textit{decreases} in output over this range. So it follows that this range coincides with the region of diminishing total returns identified in part a.

In part c, we saw that the marginal product curve was increasing over the range \( 0 < L < 200 \). And this is precisely the range identified as the region of increasing marginal returns in part a.
And finally, in part c, we saw that the marginal product curve peaked at $L = 200$ and was then decreasing over the range $200 < L < 500$. And this is precisely the range identified as the region of diminishing marginal returns in part a.

4. Ch 6, Problem 6.12

Suppose the production function is given by the following equation (where $a$ and $b$ are positive constants): $Q = aL + bK$. What is the marginal rate of technical substitution of labor for capital ($MRTS_{L,K}$) at any point along an isoquant?

For this production function $MP_L = a$ and $MP_K = b$. The $MRTS_{L,K}$ is therefore

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{a}{b}$$

5. For each of the following production functions, graph a typical isoquant and determine whether the marginal rate of technical substitution of labor for capital ($MRTS_{L,K}$) is diminishing, constant, increasing, or none of these.

a. $Q = LK$

![Graph of Q = LK](image)

Since the isoquants are bowed towards the origin, $MRTS_{L,K}$ is diminishing.

b. $Q = L\sqrt{K}$
Since the isoquants are bowed towards the origin, $\text{MRTS}_{l,k}$ is diminishing.
c. \[ Q = L^{2/3} K^{1/3} \]

Since the isoquants are bowed towards the origin, \( MRTS_{l,k} \) is diminishing.

d. \[ Q = 3L + K \]

Since the isoquants are straight lines, \( MRTS_{l,k} \) is constant.
e. \( Q = \min\{3L, K\} \)

\[ MRTS_{L,K} = \infty \] along the vertical portion of the isoquant and 0 along the horizontal portion of the isoquant; it is undefined at the “elbow point.” So, although \( MRTS_{L,K} \) decreases from \( \infty \) to 0 as we pass through the “elbow point” of the isoquant, it is not decreasing everywhere: \( MRTS_{L,K} \) is constant along the vertical leg and constant along the horizontal leg. Therefore, the answer is none of these.