Modal Analysis of Undamped Forced MDOF Systems
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§21.1. Introduction

The modal analysis method presented in the previous Lecture for unforced, undamped, MDOF dynamical systems, is extended here to the case of applied forces as well as prescribed base motions. It will be seen that extensions are relatively minor as long as the system is undamped. The main new concept required is that of modal forces. This ingredient makes the modal equations non-homogeneous, and generally requires computation of both homogeneous (transient) and particular (steady-state) solutions. (In the unforced case, the homogeneous solution is sufficient.)

If the prescribed force or base motion is harmonic, a comparison of the response amplitude to that of the static response will allow us to introduce the frequency response functions for MDOF. This generalizes the SDOF results of Lecture 17. The FRF provide a quick visualization of resonances and antiresonances. The latter is a new phenomenon that only arises in MDOF systems.

§21.2. Modal Analysis of Forced Undamped MDOF System

We again consider the two-DOF example of §22.1. As numerical values we take

\[
\begin{align*}
m_1 &= 2, \quad m_2 = 1, \quad c_1 = c_2 = 0, \quad k_1 = 6, \quad k_2 = 3, \quad p_1 = p_1(t), \quad p_2 = p_2(t). \quad (21.1)
\end{align*}
\]

![Figure 21.1. Two-DOF, forced, undamped spring-mass example system: (a) configuration, (b) DFBD.](image)
where \( p_1(t) \) and \( p_2(t) \) are specified force histories applied to masses \( m_1 \) and \( m_2 \), respectively. The resulting undamped but forced system is displayed in Figure 21.1. The known matrices and vectors in the general MDOF equation of motion (EOM): \( \mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{p} \), become

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
9 & -3 \\
-3 & 3 \\
\end{bmatrix}, \quad \begin{bmatrix}
p_1(t) \\
p_2(t) \\
\end{bmatrix}.
\]

(21.2)

and the matrix EOM becomes

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\end{bmatrix} + \begin{bmatrix}
9 & -3 \\
-3 & 3 \\
\end{bmatrix} \begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\end{bmatrix} = \begin{bmatrix}
p_1(t) \\
p_2(t) \\
\end{bmatrix}.
\]

(21.3)

Comparing to the EOM in §23.1, it is seen that the only difference is the presence of specified forces on the RHS. As a result the general solution is the sum of the homogeneous and particular solutions. The modal analysis techniques described there can be reused with only minor modifications described next.

§21.2.1. Modal Forces

It is convenient to write a system such as (21.2) in the compact matrix form

\[
\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t) = \mathbf{p}(t).
\]

(21.4)

Suppose that the modal eigenanalysis of the unforced version of (21.4), that is, \( \mathbf{M} \ddot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{0} \), has been carried out as described in the previous two Lectures, and that the modal matrix \( \Phi \) has been constructed with the mass-orthogonalized eigenvectors \( \phi_i \) as columns. Modal analysis relies on the assumption introduced in §23.1.5 of Lecture 23:

\[
\mathbf{u}(t) = \Phi \eta(t),
\]

(21.5)

where \( \eta(t) \) is the vector of modal amplitudes \( \eta_i(t) \). (The relation (21.5) is also called modal superposition, modal decomposition, or spectral decomposition in the literature.) Because \( \Phi \) does not depend on time, \( \dot{\mathbf{u}} = \Phi \dot{\eta}(t) \) and \( \ddot{\mathbf{u}} = \Phi \ddot{\eta}(t) \). Insert these into (21.4) and then premultiply through by \( \Phi^T \) to get

\[
\Phi^T \mathbf{M} \Phi \ddot{\eta}(t) + \Phi^T \mathbf{K} \Phi \dot{\eta}(t) = \Phi^T \mathbf{p}(t).
\]

(21.6)

But \( \Phi^T \mathbf{M} \Phi = \mathbf{M}_g \) and \( \Phi^T \mathbf{K} \Phi = \mathbf{K}_g \) are the generalized mass and stiffness matrices, respectively, which are diagonal. Furthermore if the eigenvectors stacked as columns of \( \Phi \) have been mass-orthonormalized as stated, \( \mathbf{M}_g = \mathbf{I} \), which is the identity matrix, whereas \( \mathbf{K}_g = \text{diag}(\omega_i^2) \) has the squared frequencies in its diagonal. To accomodate the RHS of (21.6) we introduce the definition

\[
\mathbf{f}(t) = \Phi^T \mathbf{p}(t).
\]

(21.7)

The entries of \( \mathbf{f}(t) \), denoted by \( f_i(t) \), are called modal forces and also (in a wider context) generalized forces. To show how those forces are obtained, consider the example system, in which the modal matrix was displayed in §23.1.5 of Lecture 23. Applying (21.7) yields

\[
\begin{bmatrix}
f_1(t) \\
f_2(t) \\
\end{bmatrix} = \Phi^T \mathbf{p}(t) = \begin{bmatrix}
\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
\end{bmatrix} \begin{bmatrix}
p_1(t) \\
p_2(t) \\
\end{bmatrix} = \begin{bmatrix}
\frac{p_1(t) + 2p_2(t)}{\sqrt{6}} \\
\frac{p_1(t) - p_2(t)}{\sqrt{3}} \\
\end{bmatrix}.
\]

(21.8)
§21.2 MODAL ANALYSIS OF FORCED UNDAMPED MDOF SYSTEM

§21.2.2. Modal Equations of Motion

Substitute (21.7) into (21.6), and account for the diagonal configuration of \( M_g = I \) and \( K_g = \text{diag}(\omega^2) \) noted above. The matrix EOM in modal coordinates becomes

\[
\ddot{\eta}(t) + \text{diag}(\omega^2) \eta(t) = f(t).
\]  

(21.9)

These are called the modal EOM. For a \( n \)-DOF dynamical system governed by the physical EOM (21.4), (21.9) represents a set of \( n \) uncoupled, nonhomogeneous equations. To display this decoupling clearly, consider the example system. Using results of the previous Lecture as well as (21.8), we obtain the modal EOM in matrix form as

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_1(t) \\
\dot{\eta}_2(t)
\end{bmatrix} +
\begin{bmatrix}
3/2 & 0 \\
0 & 6
\end{bmatrix}
\begin{bmatrix}
\eta_1(t) \\
\eta_2(t)
\end{bmatrix} =
\begin{bmatrix}
f_1(t) \\
f_2(t)
\end{bmatrix},
\]  

(21.10)

which uncouples into two nonhomogeneous, second-order ODEs:

\[
\begin{align*}
\ddot{\eta}_1(t) + (3/2) \eta_1(t) &= f_1(t) = (p_1(t) + 2p_2(t))/\sqrt{6}, \\
\ddot{\eta}_2(t) + 6 \eta_2(t) &= f_2(t) = (p_1(t) - p_2(t))/\sqrt{3}.
\end{align*}
\]

(21.11)

The solution of each ODE is the sum of its homogeneous solution, which depends on the initial conditions (which can be obtained by the method presented in §23.1.7 of Lecture 23) and the particular solution, which depends on the forcing terms \( f_1(t) \) and \( f_2(t) \).* Once the modal solutions are on hand, they can be transformed to physical coordinates via \( u = \Phi \eta \).

§21.2.3. Forced Response Example

To illustrate the analysis process for the example problem, a harmonic force of amplitude \( F_2 \), circular frequency \( \Omega \), and cosine dependency, acting on mass 2, plus I.C. as indicated:

\[
p(t) = \begin{bmatrix} 0 \\ F_2 \cos \Omega t \end{bmatrix}, \quad u_0 = \begin{bmatrix} * \\ * \end{bmatrix}, \quad v_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

(21.12)

Here \( F_2 \) and \( \Omega \) will be kept as variables during the analysis to help the construction of frequency response function (FRF) graphs later. The initial conditions on displacements are “cooked up” so that all responses come out proportional to \( \cos \Omega t \) only.† The modal forces are \( f_1(t) = (2/\sqrt{6}) F_2 \cos \Omega t \) and \( f_1(t) = -(1/\sqrt{3}) F_2 \cos \Omega t \). The modal equations are

\[
\begin{align*}
\ddot{\eta}_1(t) + (3/2) \eta_1(t) &= (2/\sqrt{6}) F_2 \cos \Omega t, \\
\ddot{\eta}_2(t) + 6 \eta_2(t) &= -(1/\sqrt{3}) F_2 \cos \Omega t.
\end{align*}
\]

(21.13)

Only the particular solution is of interest. To find that quickly, note that a particular solution of \( \ddot{\eta} + \omega^2 \eta = A \cos \Omega t \) is‡ \( \eta_p = A \cos \Omega t/(\omega^2 - \Omega^2) \). Thus

\[
\begin{align*}
\eta_1(t) &= (2/\sqrt{6}) F_2 \cos \Omega t, \\
\eta_2(t) &= - (1/\sqrt{3}) F_2 \cos \Omega t.
\end{align*}
\]

(21.14)

* Recall from Lecture 21 that this are called “transient” and “steady state” solutions, respectively, by structural engineers when the system includes damping.
† Else a \( \cos \omega t \) term will appear in the response functions.
‡ To get that result, insert the guess \( \eta_{guess} = B \cos \Omega t \) in \( \ddot{\eta} + \omega^2 \eta = A \cos \Omega t \), and solve for \( B \).

21–5
Here we have kept the natural frequencies $\omega_1$ and $\omega_2$ as generic to make the configuration of the particular solutions more visible.

Combining the solutions (21.14) as per $u(t) = \Phi \eta(t)$ yields

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \frac{1}{3} F_2 \cos \Omega t \begin{bmatrix} \frac{\omega_1^2 - \Omega^2}{\omega_1^2 - \Omega^2} & -\frac{1}{\omega_1^2 - \Omega^2} \\ \frac{2}{\omega_1^2 - \Omega^2} & + \frac{1}{\omega_1^2 - \Omega^2} \end{bmatrix} \begin{bmatrix} \frac{\omega_1^2 - \Omega^2}{\omega_1^2 - \Omega^2} \\ \frac{2}{\omega_1^2 - \Omega^2} \end{bmatrix} F_2 \cos \Omega t.$$

The last expression is obtained upon substituting $\omega_1^2 = 3/2$ and $\omega_2^2 = 6$. The initial displacements are obtained by setting $t = 0$ in (21.15): $u_1(0) = 6 F_2/((6 - \Omega^2)(3 - 2\Omega^2))$ and $u_2(0) = 2 F_2(9 - 2\Omega^2)/((6 - \Omega^2)(3 - 2\Omega^2))$; those would replace the asterisks in (21.12). Velocities are obtained by direct differentiation with respect to $t$:

$$\dot{u}(t) = \begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = -\frac{F_2 \Omega \sin \Omega t}{(6 - \Omega^2)(3 - 2\Omega^2)} \begin{bmatrix} 6 \\ 2(9 - 2\Omega^2) \end{bmatrix}.$$

These vanish at $t = 0$, in agreement with (21.12).

Figure 21.2 plots displacement responses (21.15) for $F_2 = 2$, over time range $0 \leq t \leq 12$, for the following four values of the excitation frequency.

$\Omega = 0$ Force $p_2$ is static and equal to $F_2 = 2$ for all time $t$. The static deflections are $u_1 = F_2/k_1 = 1/3$ and $u_2 = F_2/k_1 + F_2/k_2 = 1$.

$\Omega = 1$ This excitation is low frequency in the sense that $\Omega < \omega_1 = \sqrt{3}/2 \approx 1.224$ and $\Omega < \omega_2 = \sqrt{6} \approx 2.449$. The masses oscillate in phase.

$\Omega = 3/\sqrt{2}$ This excitation, with numerical value $\Omega \approx 2.12132$, is an intermediate frequency in the sense that $\Omega > \omega_1 = \sqrt{3}/2 \approx 1.224$ and $\Omega < \omega_2 = \sqrt{6} \approx 2.449$. Under this particular $\Omega$ it may be verified that $u_2(t) = 0$ for all time since the second entry of the rightmost vector in (21.15) is identically zero. Thus mass 2 never moves. This phenomenon is called antiresonance.**

$\Omega = 3$ This excitation is high frequency in the sense that $\Omega > \omega_1 = \sqrt{3}/2 \approx 1.224$ and $\Omega > \omega_2 = \sqrt{6} \approx 2.449$. The masses oscillate at $180^\circ$ out of phase, moving in opposite directions.

Note that the vertical plot range: $-3$ to $+3$ is the same for all four plots in Figure 21.2, to facilitate visual amplitude comparison.

§21.2.4. Frequency Response Functions

Recall from Lecture 18 that a Frequency Response Function (FRF) is the ratio $D_s$ of the steady-state displacement response amplitude to the static displacement, computed as a function of excitation frequency. The FRF, also called steady-state magnification factor, pinpoints at a glance the location

** Antiresonance has important technical applications, such as vibration isolation and noise mitigation.
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Figure 21.2. Response of harmonically excited example system for 4 values of the excitation frequency $\Omega$: (a) $\Omega = 0$, a static force, (b) $\Omega = 1$, below both natural frequencies: masses oscillate in phase; (c) $\Omega = \frac{3}{\sqrt{2}} \approx 2.12132$, which produces mass-2 antiresonance; and (d) $\Omega = 3$, higher than both natural frequencies: masses oscillate $180^\circ$ out of phase.

Figure 21.3. Frequency Response Functions (FRF) of harmonically excited example system: (a) natural scale plot, (b) log-log plot.

of resonances. The log-log version of this plot is obtained by taking the absolute value of the
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For a MDOF system, there are as many FRF as DOF. For our example the static displacements are 
\[ u_{s1} = F_2 \cos \Omega t / k_1 = F_2 \cos \Omega t / 6 \] and 
\[ u_{s2} = F_2 \cos \Omega t (1/k_1 + 1/k_2) = F_2 \cos \Omega t / 2. \] (These values may be verified by making \( \Omega = 0 \) in (21.15).) The amplitude of the steady state response is obtained by setting \( \cos \Omega t = 1 \) in (21.15), which gives 
\[ U_1 = 6F_2/(6 - \Omega^2)(3 - 2\Omega^2) \] and 
\[ U_2 = 2F_2(9 - 2\Omega^2)/(6 - \Omega^2)(3 - 2\Omega^2). \] Consequently the magnification factors at the point mass locations are

\[ D_{s1} = \frac{U_1}{u_{s1}} = \frac{18}{(6 - \Omega^2)(3 - 2\Omega^2)}, \quad D_{s2} = \frac{U_2}{u_{s2}} = \frac{2(9 - 2\Omega^2)}{(6 - \Omega^2)(3 - 2\Omega^2)}. \] (21.17)

Figure 21.3(a) plots the FRF for masses 1 and 2 as the excitation frequency \( \Omega \) sweeps over the range \( 0 \leq \Omega \leq 5 \). Figure 21.3(b) is a log-log plot version that covers a wider excitation spectrum. This version pinpoints resonances and the antiresonance better.

§21.3. MDOF Under Prescribed Base Motion

The two-DOF example problem is taken to be unforced, as in the previous Lecture, but under prescribed base motion \( u_b(t) \). See Figure 21.4(a).
§21.4. Base Motion EOM

Using the DFBD shown in Figure 21.4(b) we construct the matrix EOM

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_1 \\
  \ddot{u}_2
\end{bmatrix}
+ \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
= \begin{bmatrix}
  k_1 u_b(t) \\
  0
\end{bmatrix}.
\]  \tag{21.18}

As specific values we take those used in previous cases:

\[
m_1 = 2, \quad m_2 = 1, \quad c_1 = c_2 = 0, \quad k_1 = 6, \quad k_2 = 3.
\]  \tag{21.19}

The base motion, however, is for now left generic as well as \(k_1\) on the RHS, so (21.18) becomes

\[
\begin{bmatrix}
  2 & 0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_1 \\
  \ddot{u}_2
\end{bmatrix}
+ \begin{bmatrix}
  9 & -3 \\
  -3 & 3
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
= \begin{bmatrix}
  k_1 u_b(t) \\
  0
\end{bmatrix}.
\]  \tag{21.20}

§21.4.1. Modal Analysis

Since the LHS of (21.20) is the same as before, the modal analysis procedure described in S24.1 for forced case applied with only cosmetic changes, since the prescribed base motion is equivalent to a force \(p_1(t) = k_1 u_b(t)\) on mass 1 whereas \(p_2(t) = 0\). In particular, the previously computed natural frequencies and mass-orthogonalized mode shapes, as well as the modal matrix formed with the latter, can be reused without change. Consequently the result (21.12) for the modal forces is applicable. Replacing there \(p_1(t) = k_1 u_b(t)\) and \(p_2(t) = 0\) gives the modal forces

\[
\begin{bmatrix}
  f_1(t) \\
  f_2(t)
\end{bmatrix}
= \Phi^T \mathbf{p}(t) = \begin{bmatrix}
  \frac{p_1(t) + 2p_2(t)}{\sqrt{6}} \\
  \frac{p_1(t) - p_2(t)}{\sqrt{3}}
\end{bmatrix}
= k_1 u_b(t) \begin{bmatrix}
  \frac{1}{\sqrt{6}} \\
  \frac{1}{\sqrt{3}}
\end{bmatrix}
\]  \tag{21.21}

The modal equations, leaving \(\omega_1\) and \(\omega_2\) symbolic for the moment, are

\[
\ddot{\eta}_1(t) + \omega_1^2 \eta_1(t) = \frac{1}{\sqrt{6}} k_1 u_b(t), \quad \ddot{\eta}_2(t) + \omega_2^2 \eta_2(t) = \frac{1}{\sqrt{3}} k_1 u_b(t).
\]  \tag{21.22}

To show solutions need a specific form for the RHS. To facilitate comparison with the forced problem, we take a harmonically varying base motion of excitation frequency \(\Omega\), amplitude \(U_b\), and cosine dependence, whence \(u_b(t) = U_b \cos \Omega t\) and the modal EOM (21.22) become

\[
\ddot{\eta}_1(t) + \omega_1^2 \eta_1(t) = \frac{1}{\sqrt{6}} k_1 U_b \cos \Omega t, \quad \ddot{\eta}_2(t) + \omega_2^2 \eta_2(t) = \frac{1}{\sqrt{3}} k_1 U_b \cos \Omega t.
\]  \tag{21.23}

We will assume that the I.C. at \(t = 0\) are such that all responses are proportional to \(\cos \Omega t\), as was done in §21.2.3. Modal solutions then reduce to a simplified form of the particular, a.k.a. steady-state, component. The steady-state solutions are

\[
\eta_1(t) = \frac{(1/\sqrt{6})}{\omega_1^2 - \Omega^2} k_1 U_b \cos \Omega t, \quad \eta_2(t) = \frac{(1/\sqrt{3})}{\omega_2^2 - \Omega^2} k_1 U_b \cos \Omega t.
\]  \tag{21.24}
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\[ u(t) = 0 \]

\[ u(t) = 1 \]

\[ u(t) = 0 \]

\[ u(t) = 1 \]

\[ u(t) = 0 \]

\[ u(t) = 1 \]

\[ u(t) = 0 \]

\[ u(t) = 1 \]

\[ u(t) = 0 \]

\[ u(t) = 1 \]

\[ u(t) = 0 \]

\[ u(t) = 1 \]

**Figure 21.5.** Response of harmonically base-excited example system for 4 values of the excitation frequency \( \Omega \): (a) \( \Omega = 0 \), a static force, (b) \( \Omega = 1 \), below both natural frequencies: masses oscillate in phase; (c) \( \Omega = \sqrt{3} \approx 1.732 \), which produces mass-1 antiresonance; and (d) \( \Omega = 3 \), higher than both natural frequencies: masses oscillate 180° out of phase.

Combining these solutions as per \( u(t) = \Phi \eta(t) \), and finally replacing \( k_1 = 6, \omega_1^2 = \frac{3}{2} \) and \( \omega_2^2 = 6 \) yields

\[
\begin{align*}
u(t) &= \left[ \begin{array}{c} u_1(t) \\ u_2(t) \end{array} \right] = \frac{1}{6} k_1 U_b \cos \Omega t \left[ \begin{array}{c}
\frac{2}{\omega_1^2 - \Omega^2} + \frac{1}{\omega_2^2 - \Omega^2} \\
\frac{1}{\omega_1^2 - \Omega^2} - \frac{1}{\omega_2^2 - \Omega^2}
\end{array} \right] \\
&= U_b \cos \Omega t \left[ \begin{array}{c}
\frac{2}{\omega_1^2 - \Omega^2} + \frac{1}{\omega_2^2 - \Omega^2} \\
\frac{1}{\omega_1^2 - \Omega^2} - \frac{1}{\omega_2^2 - \Omega^2}
\end{array} \right] = \frac{6U_b}{(6 - \Omega^2)(3 - 2\Omega^2)} \left[ \begin{array}{c}
2 - \Omega^2 \\
3 - \Omega^2
\end{array} \right].
\end{align*}
\]

Figure 21.5 plots displacement responses (21.24) for base amplitude motion \( U_b = 1 \), over time range \( 0 \leq t \leq 12 \), for the following four values of the excitation frequency.

**\( \Omega = 0 \)** Base motion is static and equal to \( U_b = 1 \) for all time \( t \). The static deflections are \( u_1 = u_2 = U_b = 1 \), since the system moves as a rigid body. This can be readily verified by setting \( \Omega = 0 \) in the last of (21.25).

**\( \Omega = 1 \)** This excitation is low frequency in the sense that \( \Omega < \omega_1 = \sqrt{3/2} \approx 1.224 \) and...
\section*{§21.4 BASE MOTION EOM}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{frequency_response.png}
\caption{Frequency Response Functions (FRF) of harmonically base-excited example system: (a) natural scale plot, (b) log-log plot.}
\end{figure}

\[ \Omega < \omega_2 = \sqrt{6} \approx 2.449. \] The two masses oscillate in phase.

\[ \Omega = \sqrt{3} \] This excitation, with numerical value \( \Omega \approx 1.732 \), is intermediate frequency in the sense that \( \Omega > \omega_1 = \sqrt{3/2} \approx 1.224 \) and \( \Omega < \omega_2 = \sqrt{6} \approx 2.449 \). Under this particular \( \Omega \) it may be verified from (21.25) that \( u_1(t) = 0 \) for all time, i.e., mass 1 never moves. As noted in §24.1.3, this phenomenon is called antiresonance.

\[ \Omega = 3 \] This excitation is high frequency in the sense that \( \Omega > \omega_1 = \sqrt{3/2} \approx 1.224 \) and \( \Omega > \omega_2 = \sqrt{6} \approx 2.449 \). The masses oscillate at 180° out of phase, moving in opposite directions.

Note that the vertical range: \(-4\) to \(+4\), is the same for all four plots in Figure 21.5 so as to facilitate visual amplitude comparison.

\section*{§21.4.2. Frequency Response Functions}

The steady-state amplification factors \( D_{s1} \) and \( D_{s2} \) are the ratio of the amplitudes of \( u_1(t) \) and \( u_2(t) \) to their static values. As noted above, the latter are \( u_{s1} = u_{s2} = U_b = 1 \). The amplitudes can be obtained from (21.25) by setting \( \cos \Omega t = 1 \). Hence

\begin{align*}
D_{s1} &= \frac{3 - \Omega^2}{(6 - \Omega^2)(3 - 2\Omega^2)}, \\
D_{s2} &= \frac{3}{(6 - \Omega^2)(3 - 2\Omega^2)}. \tag{21.26}
\end{align*}

The graph of \( D_{si} \) versus excitation frequency \( \Omega \) are called the frequency response plots or FRF. For this example they are shown in Figure 21.6. Figure 21.6(a) shows the two FRF for masses 1 and 2 as \( \Omega \) sweeps over the range \( 0 \leq \Omega \leq 5 \). Figure 21.6(b) is a log-log plot version that covers a wider excitation spectrum; in this case the absolute value of the amplitude is taken. The log-log displays both resonances and the antiresonance better.