Modal Analysis of MDOF Unforced Undamped Systems
# Lecture 20: Modal Analysis of MDOF Unforced Undamped Systems

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§20.1. Introduction

This subject of this Lecture and of the next one is modal analysis. This is a technique by which the equations of motion (EOM), which are originally expressed in physical coordinates, are transformed to modal coordinates using the eigenvalues and eigenvectors gotten by solving the undamped frequency eigenproblem. The transformed equations are called modal equations. In a mathematical context, modal coordinates are also called generalized coordinates, or principal coordinates. For structural dynamics they can be interpreted as response amplitudes of orthonormalized vibration modes.

The distinguishing feature of modal equations is that for an undamped system they uncouple. Consequently, each modal equation may be solved independently of the others. Once computed, modal responses may be transformed back to physical coordinates and superposed to produce the physical response of the original system.

The method is a particular case of what is known in applied mathematics as orthogonal projection methods. Instead of tackling the original equations directly, they are projected onto another space (the 'modal space' in the case of dynamics) in which equations decouple.

§20.2. Modal Analysis of Unforced Undamped MDOF System

Here we retake the two-DOF example introduced in §19.1. As numerical values we take* 

\[ m_1 = 2, \quad m_2 = 1, \quad c_1 = c_2 = 0, \quad k_1 = 6, \quad k_2 = 3, \quad p_1 = p_2 = 0. \]  

(20.1)

The resulting undamped and unforced system is displayed in Figure 20.1. The known matrices and vectors in the general EOM: \( \mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{p} \), become

\[
\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]  

(20.2)

and the equations become

\[
\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} + \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. 
\]  

(20.3)

§20.2.1. Natural Frequencies

The free-vibration eigenproblem associated with (20.3) is

\[
\begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 9 - 2\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. 
\]  

(20.4)

The characteristic polynomial equation is

\[
\det \begin{bmatrix} 9 - 2\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix} = 18 - 15\omega^2 + 2\omega^4 = (3 - 2\omega^2)(6 - \omega^2) = 0.
\]  

(20.5)

The roots of (20.5) give the two squared frequencies

\[
\omega_1^2 = \frac{3}{2} = 1.5, \quad \omega_2^2 = 6.
\]  

(20.6)

* Note that no units are specified; being tacitly understood that a consistent set of physical units, SI or English, is used throughout.
§20.2.2. Vibration Modes

To get $U_1$, replace $\omega_1^2 = 3/2$ into the second of (20.4), set its first entry $U_{11}$ to one, and solve for the second entry $U_{12}$:

$$
\begin{bmatrix}
9-2 \times (3/2) & -3 \\
-3 & 3-3/2
\end{bmatrix}
\begin{bmatrix}
U_{11} \\
U_{12}
\end{bmatrix} =
\begin{bmatrix}
6 & -3 \\
-3 & 3/2
\end{bmatrix}
\begin{bmatrix}
U_{11} \\
U_{12}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix} \Rightarrow U_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.
$$

(20.7)

To get $U_2$, replace $\omega_2^2 = 6$ into the second of (20.4), set its first entry $U_{21}$ to one, and solve for the second entry $U_{22}$:

$$
\begin{bmatrix}
9-2 \times 6 & -3 \\
-3 & 3-6
\end{bmatrix}
\begin{bmatrix}
U_{21} \\
U_{22}
\end{bmatrix} =
\begin{bmatrix}
-3 & -3 \\
-3 & -3
\end{bmatrix}
\begin{bmatrix}
U_{21} \\
U_{22}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix} \Rightarrow U_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
$$

(20.8)

We now apply the first normalization method described in §19.3.3, by forcing the largest entry to be +1:

$$
\phi_1 = \frac{1}{2} U_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \quad \phi_2 = U_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
$$

(20.9)

The vibration modes are pictured in Figure 20.2. Note that in the first mode the masses oscillate in phase while in the second one they are $180^\circ$ out of phase, moving opposite to each other.

It is convenient at this point to verify the orthogonality property: $\phi_i^T M \phi_j = 0$ and $\phi_i^T K \phi_j = 0$ for $i \neq j$. Here only one combination, namely $i = 1$ and $j = 2$, needs to be tested:

$$
\phi_1^T M \phi_2 = \begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0, \quad \phi_1^T K \phi_2 = \begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0.
$$

(20.10)
20.2 MODAL ANALYSIS OF UNFORCED UNDAMPED MDOF SYSTEM

Mode 1, $\omega^2_1 = 3/2$

Mode 2, $\omega^2_2 = 6$

Red lines mark the static equilibrium positions of the masses.

Figure 20.2. Vibration modes for example system.

Note that there is no need to explicitly check that $\phi^T_1 M \phi_1 = 0$ and $\phi^T_2 K \phi_1 = 0$ because $(\phi^T_1 M \phi_2)^T = \phi^T_2 M^T \phi_1 = \phi^T_2 M \phi_1$, since $M = M^T$ if $M$ is symmetric. Likewise for $K$.

The vibration modes (20.9) are orthogonal but not orthonormal with respect to the mass matrix. To achieve that property it is necessary to rescale them so that generalized masses are unity. This is done next.

20.2.3. Generalized Mass and Stiffness

Next we compute the generalized masses and stiffnesses associated with the modes (20.10):

\[
M_1 = \phi^T_1 M \phi_1 = \begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 3/2, \\
M_2 = \phi^T_2 M \phi_2 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3, \\
K_1 = \phi^T_1 K \phi_1 = \begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = 9/4, \\
K_2 = \phi^T_2 K \phi_2 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 18.
\]

Quick check:

\[
\begin{align*}
\omega^2_1 &= \frac{K_1}{M_1} = \frac{9/4}{3/2} = 3/2, \\
\omega^2_2 &= \frac{K_2}{M_2} = \frac{18}{3} = 6, \\
\end{align*}
\]

as expected. Note that should eigenvectors be normalized in a different way, $M_1, M_2, K_1, K_2$ will be generally different, but the ratios $\omega^2_i = K_i / M_i$ remain invariant.
§20.2.4. Eigenvector Mass Orthonormalization

Next we renormalize \( \phi_1 \) and \( \phi_2 \) to get unit generalized masses. These rescaled vectors will be initially denoted by \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \), respectively. After renormalization the tildes are dropped.

Suppose \( \tilde{\phi}_1 = c_1 \phi_1 \). To find \( c_1 \), insert this into \( 1 = \tilde{\phi}_1^T M \tilde{\phi}_1 = (c_1 \phi_1^T) M (c_1 \phi_1) = c_1^2 \phi_1^T M \phi_1 = c_1^2 \times 3/2 \), whence \( c_1 = 1/\sqrt{3/2} = \sqrt{2/3} \). Likewise \( c_2 = 1/\sqrt{3} \). Dropping the tildes for brevity, the mass-orthonormalized vibration modes are

\[
\phi_1 = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix} \begin{bmatrix} 0.4088 \\ 0.8165 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 1/3 \\ -1 \end{bmatrix} \begin{bmatrix} 0.5773 \\ -0.5773 \end{bmatrix}
\]

(20.13)

After this renormalization,

\[
\phi_1^T M \phi_1 = 1, \quad \phi_2^T M \phi_2 = 1, \quad \phi_1^T K \phi_1 = \frac{3}{2} = \omega_1^2, \quad \phi_2^T K \phi_2 = 6 = \omega_2^2.
\]

(20.14)

§20.2.5. Generalized Coordinates and Modal Matrix

We will express† the displacement vector \( u(t) \) in terms of normal coordinates \( \eta_1(t) \) and \( \eta_2(t) \), as follows:

\[
u = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \equiv \phi_1 \eta_1(t) + \phi_2 \eta_2(t) = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \Phi \eta.
\]

(20.15)

In a mathematical context, the \( \eta_i(t) \)'s are called generalized coordinates or principal coordinates. They represent the amplitude of the orthonormalized mode shapes, or modal amplitudes for short. Equation (20.15) is an instance of modal superposition, which is a general feature of linear dynamical systems.

Two new matrix symbols appear in (20.15). The normal coordinate vector \( \eta \) collects \( \eta_1(t) \) and \( \eta_2(t) \) as its entries. The modal matrix \( \Phi \) is formed by stacking the mass-orthogonal eigenvectors as columns:

\[
\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{6}}{2} & 1 \\ \frac{\sqrt{6}}{2} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0.4088 \\ 0.8165 \end{bmatrix} \begin{bmatrix} 0.5773 \\ -0.5773 \end{bmatrix}.
\]

(20.16)

Note that \( \dot{\eta} = \Phi \dot{\eta} \) and \( \ddot{\eta} = \Phi \ddot{\eta} \), because \( \Phi \) does not depend on time. With the help of \( \Phi \), the orthonormality conditions can be expressed compactly as‡

\[
\Phi^T M \Phi = M_g = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi^T K \Phi = K_g = \text{diag}[\omega_i^2] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 6 \end{bmatrix}.
\]

(20.17)

Here \( M_g \) and \( K_g \) denote the generalized mass matrix and generalized stiffness matrix, respectively. If \( \Phi \) is built with mass-orthogonalized eigenvectors, as in (20.16), \( M_g \) reduces to the identity matrix while \( K_g \) becomes a diagonal matrix with squared frequencies stacked along its diagonal.

† This is a consequence of the so-called expansion theorem for MDOF systems; in the applied mathematics context it is known as the spectral decomposition. See Section 10.1.9 of the Craig-Kurdila textbook for details.

‡ The Craig-Kurdila textbook uses bold italics symbols for generalized mass and stiffness matrices. Since those fonts are not available to the \LaTeX\ document processor used for these Lectures, we shall use \( g \) subscripts to denote those quantities.
§20.2.6. Modal Equations of Motion

To transform the undamped, unforced EOM: $M \ddot{u} + K u = 0$ to modal coordinates, replace $u = \Phi \eta$ and $\dot{u} = \Phi \dot{\eta}$ into it, and premultiply by $\Phi^T$:

$$
\Phi^T M \Phi \dot{\eta} + \Phi^T K \Phi \eta = \Phi^T 0 = 0.
$$

(20.18)

Using the generalized matrices introduced in (20.17) we get

$$
\ddot{\eta} + K_g \eta = 0.
$$

(20.19)

For the example problem,

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_1(t) \\
\ddot{\eta}_2(t)
\end{bmatrix}
+ 
\begin{bmatrix}
3/2 & 0 \\
0 & 6
\end{bmatrix}
\begin{bmatrix}
\eta_1(t) \\
\eta_2(t)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

(20.20)

Because the matrices in (20.20) are diagonal, (20.20) uncouples into two homogeneous, second-order ODE already in canonical form:

$$
\ddot{\eta}_1(t) + (3/2) \eta_1(t) = 0, \quad \ddot{\eta}_2(t) + 6 \eta_2(t) = 0.
$$

(20.21)

Each ODE models an unforced, undamped SDOF oscillator. As such it may be treated using the methods of Lecture 17 as long as initial conditions (IC) are known. Once solutions $\eta_1(t)$ and $\eta_2(t)$ are available, they can be combined via the mode superposition relation (20.15) to get the physical response $u(t) = \Phi \eta(t)$. But to solve (20.21) we need their IC in modal coordinates.

§20.2.7. Modal Initial Conditions

Suppose that the initial conditions for a MDOF system are

$$
u(0) = u_0, \quad \dot{u}(0) = v_0.
$$

(20.22)

Here $u_0$ and $v_0$ denote vectors of initial displacements and velocities, respectively. Because $u(t) = \Phi \eta(t)$ and $\dot{u}(t) = \Phi \dot{\eta}(t)$ for any time $t$, setting $t = 0$ we get

$$
u_0 = u(0) = \Phi \eta(0), \quad v_0 = \dot{u}(0) = \Phi \dot{\eta}(0).
$$

(20.23)

These can be solved for $\eta_0 = \eta(0)$ and $\dot{\eta}_0 = \dot{\eta}(0)$ by inverting the modal matrix:

$$
\eta_0 = \eta(0) = \Phi^{-1} u_0, \quad \dot{\eta}_0 = \dot{\eta}(0) = \Phi^{-1} v_0.
$$

(20.24)

or equivalently, solving the linear systems (20.23). But there is a more elegant scheme that only requires matrix multiplication. Postmultiplying both sides of $\Phi^T M \Phi = I$ by $\Phi^{-1}$ gives $\Phi^{-1} = \Phi^T M$, whence

$$
\eta_0 = \Phi^T M u_0, \quad \dot{\eta}_0 = \Phi^T M v_0.
$$

(20.25)

Remark 20.1. Should the eigenvectors $\phi_i$ not be mass-orthonormal, we have the more complicated relations $M_g \eta_0 = \Phi^T M u_0$ and $M_g \dot{\eta}_0 = \Phi^T M v_0$, in which $M_g = \Phi^T M \Phi$ is the generalized mass matrix. Consequently

$$
\eta_0 = M_g^{-1} \Phi^T M u_0, \quad \dot{\eta}_0 = M_g^{-1} \Phi^T M v_0.
$$

(20.26)

Since $M_g$ is always diagonal, so is $M_g^{-1}$. Consequently (20.26) merely amounts to scaling entries of $\eta_0$ and $\dot{\eta}_0$ by reciprocals of the generalized masses.

20–7
§20.3. Unforced Response Example

The following initial conditions (IC) are specified for the 2-DOF mass-spring system of Figure 20.1: mass $m_1$ is given a unit initial velocity along $+x$, while all other initial values are zero. In matrix form:

$$u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (20.27)$$

The modal IC are obtained from (20.25) as

$$\eta_0 = \begin{bmatrix} \eta_{10} \\ \eta_{20} \end{bmatrix} = \Phi^T M u_0 = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\dot{\eta}_0 = \begin{bmatrix} \dot{\eta}_{10} \\ \dot{\eta}_{20} \end{bmatrix} = \Phi^T M v_0 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \sqrt{6} \\ \frac{2}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0.8165 \\ 1.1547 \end{bmatrix}. \quad (20.28)$$

The solution of the modal equations (20.20) for the initial conditions (20.28) are

$$\eta_1(t) = \frac{\dot{\eta}_{10}}{\omega_1} \sin \omega_1 t = \frac{2}{3\sqrt{3}} \sin(\sqrt{6} t) = \frac{2}{3} \sin(\sqrt{3/2} t) = 0.6667 \sin(1.2247 t),$$

$$\eta_2(t) = \frac{\dot{\eta}_{20}}{\omega_2} \sin \omega_2 t = \frac{2/\sqrt{3}}{\sqrt{6}} \sin(\sqrt{6} t) = \frac{2}{3\sqrt{2}} \sin(\sqrt{6} t) = 0.4714 \sin(2.4495 t). \quad (20.29)$$

These can be transformed back to physical coordinates via the modal matrix, as per $u(t) = \Phi \eta(t)$. Here are the calculations for the mass displacement responses, with numerical expressions listed...
to 4-place accuracy:

\[
\begin{bmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\begin{bmatrix}
\frac{2}{3} \sin(\sqrt{3/2} t) \\
\frac{2}{3\sqrt{2}} \sin(\sqrt{6} t)
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{3} \sqrt{3/2} \left( \sin(\sqrt{3/2} t) + \sin(\sqrt{6} t) \right) \\
\frac{1}{3} \sqrt{3/2} \left( 2 \sin(\sqrt{3/2} t) - \sin(\sqrt{6} t) \right)
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.2722 \left( \sin(1.2247t) + \sin(2.4495t) \right) \\
0.2722 \left( 2 \sin(1.2247t) - \sin(2.4495t) \right)
\end{bmatrix}
= \begin{bmatrix}
0.3333 \left( \cos(1.2247t) + 2 \cos(2.4495t) \right) \\
0.6667 \left( \cos(1.2247t) - \cos(2.4495t) \right)
\end{bmatrix}
\]

Mass velocities are obtained by direct numerical differentiation with respect to \( t \):

\[
\begin{bmatrix}
\dot{u}_1(t) \\
\dot{u}_2(t)
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{3} \left( \cos(\sqrt{3/2} t) + 2 \cos(\sqrt{6} t) \right) \\
\frac{2}{3} \left( \cos(\sqrt{3/2} t) - \cos(\sqrt{6} t) \right)
\end{bmatrix}
\]

The results (20.30) and (20.31) are plotted in Figure 20.3 over \( 0 \leq t \leq 12 \). Since there is no damping, the harmonic oscillation patterns visible in the Figure will repeat with no distortions or amplitude decay. In other words, the free vibrations will go on forever.* Observe that the initial conditions (20.27) are correctly reproduced.

---

* The reason behind the repeating oscillation patterns in Figure 20.3 is that frequency \( \omega_2 \) is exactly twice \( \omega_1 \) in the example.