Harmonically Forced SDOF Oscillator
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§18.1. Introduction

This Lecture generalizes the solutions presented in Lecture 17 for the SDOF oscillator by considering an applied harmonic force for \( t \geq 0 \). The frequency of this harmonic force is called the exciting frequency. This forced SDOF configuration is called the forced harmonic oscillator.

When an external force is applied, the EOM, while still a second-order ODE, becomes nonhomogeneous. The total response is the superposition of the homogeneous solution, which is defined by the initial conditions, and the particular solution, which is defined by the prescribed harmonic force. Structural engineers call these two components the starting transient or simply transient, and the steady-state components of the dynamic response, respectively.

Those names suggest that the particular (= steady-state) solution will eventually dominate. And indeed if even a slight amount of damping is present, the homogenous component becomes negligible after sufficient time has elapsed, and we can focus our attention on the particular or steady-state solution. This focus will allow us to exhibit the important phenomenon of resonance. This term quantifies the response amplification that occurs when the exciting frequency approaches the natural frequency of the oscillator.

We will consider a damped oscillator from the start since the presence of damping is crucial in establishing the peak magnitude of resonance amplification. The undamped case can be easily recovered by setting the damping factor to zero.

\[ p(t) = p_0 \cos \Omega t, \quad \text{(18.1)} \]

**Figure 18.1.** Damped spring-dashpot-mass oscillator under prescribed harmonic force excitation: (a) dynamic equilibrium position; (b) dynamic FBD.

§18.2. Harmonically Excited Viscous-Damped SDOF Oscillator

We consider again the mass-dashpot-spring system studied in §17.3, but now subjected to a harmonic excitation force

\[ p(t) = p_0 \cos \Omega t, \]
in which $\Omega$ is the excitation frequency. See Figure 18.1(a). Note that all static actions, such as gravity, are omitted from the drawing to improve clarity, being tacitly understood that $u = u(t)$ is the dynamic motion measured from the static equilibrium position.

§18.2.1. Equation of Motion and Steady-State Solution

The DFD shown in Figure 18.1(b) gives us the dynamic equations

$$m \ddot{u} + c \dot{u} + ku = p_0 \cos \Omega t.$$  \hspace{1cm} (18.2)

As remarked in the Introduction, the complete response will be the sum of the transient (= homogeneous) and steady-state (= particular) components. If damping is present, after a while the transient part, which is that dependent on initial conditions, becomes unimportant. This solution, denoted by $u_p(t)$, with subscript $p$ for “particular,” is conveniently expressed in the phased form

$$u_p(t) = U \cos(\Omega t - \alpha).$$  \hspace{1cm} (18.3)

The associated velocity and acceleration are

$$\dot{u}_p(t) = -\Omega U \sin(\Omega t - \alpha), \quad \ddot{u}_p(t) = -\Omega^2 U \cos(\Omega t - \alpha).$$  \hspace{1cm} (18.4)

§18.2.2. Magnification Factor and Phase Lag

Inserting this into the EOM (18.2), and representing its components as a force vector polygon in the phase space (see Craig-Kurdila, pages 87–88) we obtain

$$(k U - m \Omega^2 U)^2 + (c \Omega U)^2 = p_0^2, \quad \tan \alpha = \frac{c \Omega}{k - m \Omega^2}. \hspace{1cm} (18.5)$$

For convenience introduce the static displacement $U_0$ and the frequency ratio as

$$U_0 = \frac{p_0}{k}, \quad r = \frac{\Omega}{\omega_n}. \hspace{1cm} (18.6)$$
§18.2 HARMONICALLY EXCITED VISCOUS-DAMPED SDOF OSCILLATOR

![Graph](image)

**Figure 18.3.** Frequency response log plots for harmonically forced viscous-damped oscillator: (a) plot of magnification factor versus frequency ratio; (b) plot of phase lag angle versus frequency ratio.

Observe that $U_0$ is the displacement that the mass would undergo if a force of magnitude $p_0$ were to be applied statically.

With the help of (18.6) the the expressions in (18.5) can be compactly expressed as

$$D_s(r) \overset{\text{def}}{=} \frac{U(r)}{U_0} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}, \quad \tan \alpha(r) = \frac{2\xi r}{1 - r^2}. \quad (18.7)$$

Here $D_s$ is the steady-state magnification factor, also called gain in control systems. These are plotted in Figures 18.2(a,b). The peaks observed as $r$ is near unity and $\xi << 1$ identify resonance.

Since $D_s$ may be quite large near resonance whereas $r$ may cover a broad range of frequencies, the plots are often shown in log-log format, as shown in Figure 18.3(a,b). The log-log frequency plot is known in control applications as the Bode plot.

§18.2.3. Frequency Response Features

The combination of amplitude versus frequency and phase versus frequency information is called the frequency response of the system. Graphs such as shown in Figures 18.2 and 18.3 are called frequency response plots. The following significant features can be observed:

1. The steady state motion (18.3) is also harmonic and has the same frequency $\Omega$ as the excitation.
2. The amplitude of the steady-state response is a function of both the amplitude and frequency of the excitation, as well as of the natural frequency and damping factor of the system. The magnification factor $D_s$ may be considerably greater than unity (near resonance) or may be less than unity.
3. The excitation $p_0 \cos \Omega t$ and the steady-state response $u_p = U \cos(\Omega t - \alpha)$ are not in phase; meaning that they do not attain their maximum values at the same times.
§18.2.4. Total Response

Transcribing results of Lecture 17 for the free damped oscillator, the total response may be expressed as

\[ u(t) = U_0 D_s \cos(\Omega t - \alpha) + u_h(t), \]

in which \( u_h(t) = e^{-\xi \omega_n t} \left( A_1 \cos \omega_n t + A_2 \sin \omega_n t \right) \).

(18.8)

where \( D_s \) and \( \alpha \) are given by (18.7) whereas \( A_1 \) and \( A_2 \) in the homogeneous solution \( u_h(t) \) are determined by the initial conditions. Since \( u_h(t) \) dies out with time if \( \xi > 0 \), that justifies the name starting transient in commonly used by structural (and electrical) engineers.

§18.3. Response to Harmonic Base Excitation

Suppose now that the SDOF oscillator is not subject to an applied force, but instead the base displaces by a prescribed harmonic motion specified in the form

\[ u_b(t) = U_b \cos(\Omega t - \alpha), \]

(18.9)

See Figure 18.4(a). This kind of excitation is commonly called ground motion or ground excitation in Civil structures subject to seismic loads. It also appears in vibration isolation design problems in which equipment, instruments or payloads must be protected from base vibratory motions caused by rocket launches, aircraft maneuvers, etc. It is also the configuration pertinent to vibration resonance tests such as the “Airplane Shaker” Lab 3 demos on November 7.

Assume that the only excitation is base motion. From the FBD of Figure 18.4(b) we get \( F_I + F_s + F_d = 0 \), or

\[ m\ddot{u} + k(u - u_b) + c(\dot{u} - \dot{u}_b) = 0, \]

(18.10)

Passing the prescribed motion terms to the RHS we arrive at the EOM:

\[ m\ddot{u} + c\dot{u} + ku = ku_b + c\dot{u}_b. \]

(18.11)
If the base excitation is harmonic as in (18.9),

\[ m \ddot{u} + c \dot{u} + ku = U_b \left[ k \cos(\Omega t - \alpha_b) + c \Omega \sin(\Omega t - \alpha_b) \right]. \quad (18.12) \]

Comparing to (18.2) shows that this is similar to the equation of motion of a harmonically forced oscillator, except that both cosine and sine terms are present in the forcing function. If the damping vanishes: \( c = 0 \), (18.12) and (18.2) are identical if \( U_0 \) is replaced by \( kU_b \).