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Average Stress & Component Design
Average Stress and Its Uses

The previous lecture emphasized that stresses vary from point to point inside a body (structure). Thus $\sigma_{xx} = \sigma_{xx}(x,y,z)$, and likewise for all other components. Mathematically, stresses form a tensor field.

Although the field viewpoint is rigorous, it may be too elaborate if its determination requires expensive and time-consuming computations, even with the help of the computer. Often the time-pressed engineer compromises by using a simplified stress analysis process based on average stresses. The key idea is not to pass to the limit as done in the previous lecture. After appropriately cutting the body to manifest internal force(s) of interest, divide those resultants by the finite area on which they act.

This shortcut is especially useful in preliminary design of 1D structural components when internal force resultants can be quickly determined from statics by a FDB. Several illustrative examples are given later.
Safety Factor (1)

A safety factor characterizes the margin for bad things to happen. Here are some actual instances:

To quantity the foregoing statement, two definitions are introduced.

Failure mode. The structure experiences something that renders it disfunctional. "Mode" identifies what. Failure may be catastrophic: a bridge collapses, an airplane wing breaks in flight. Or something less drastic: a building foundation settles and cracks appear.

Design Load. A load system used for designing the structure and its components. For real structures there may be many such systems, identified as design load cases. These may include normal events while in service, as well as abnormal ones that the structure is supposed to survive without catastrophic effects when there are lives at stake. Examples: earthquakes, hurricanes, emergency landings, ...
Critical Safety Factor

Assume we have a design load case with forces $P_1$, $P_2$ ... and that under the given loads values the structure is fine. (Load cases may include applied moments, distributed loads, temperature changes, etc., but for simplicity think for now of point forces.) Suppose that the loads are proportionally increased to $sP_1$, $sP_2$ ..., where $s > 1$ is a magnification factor, and that the first failure mode triggered by this increase happens at $s_F$. This $s_F$ is called the safety factor for that load case with respect to that failure mode. If, as generally happens, there are several load cases, the process is repeated for each one. The smallest $s_F$ encountered in this sweep is the critical safety factor, which should not exceed design criteria. Obviously it should not be less than a certain target.

The foregoing description oversimplifies actual practice. For example, some loads, such as own weight, should be kept fixed. Furthermore, statistical and cost considerations come into the picture. Not all load cases may be equally probable (how often is an aircraft hit by a meteorite?), while the cost of achieving adequate safety against all possible events may be prohibitive.

A detailed safety analysis that includes statistical and cost data generally requires the use of sophisticated computer programs. For preliminary component sizing, however, a shortcut based on back-of-the-envelope stress analysis may be sufficient.
Digression: Why Component Design?

The failure of a component may trigger that of the whole structure (For want of a nail the shoe was lost ...) The well known 1986 disaster of the Challenger space shuttle, which cost $6.7B in 1971 dollars, was triggered by a $25 O-ring:
Strength Safety Factor

A simplified strength design based on average stresses and given safety factor proceeds as follows. A structural component such as a bar or beam is subject to known loads that come from an analysis of the whole structure. A failure mode of the isolated component is assumed. In the ensuing examples, that mode will represent a failure of the material when a stress level is reached.

But which stress? Here a distinction should be made between brittle and ductile materials. Brittle materials such as fiberglass or cast iron fail by the maximum tensile normal stress reaching a $\sigma_{\text{fail}}$ level. Ductile materials such as metals or alloys fail by the maximum absolute shear stress reaching a $\tau_{\text{fail}}$ level. Using FBD statics compute the average normal stresses $\sigma_{\text{avg}}$ or the average shear stress $\tau_{\text{avg}}$ over the area of an appropriate cut affected by the failure mode. The strength safety factor is the ratio

$$s_F = \frac{\sigma_{\text{fail}}}{\sigma_{\text{avg}}} \quad \text{or} \quad s_F = \frac{\tau_{\text{fail}}}{\tau_{\text{avg}}}$$

as appropriate to the material type and failure mode (more complicated failure criteria do exist, but will not be covered in this course; they are studied in ASEN 4012). For strength design $s_F$ is picked by practice, design codes or experience, and the component sized accordingly. Several design examples follow.
Ex 1: Axially Loaded Brittle Bar

Material data: cast iron, fails by normal stress at $\sigma_{\text{fail}} = 40$ ksi (kilopound/sq-in)

Failure mode: cross section fractures if axial normal stress reaches $\sigma_{\text{fail}}$

Safety factor: $s_F = 8$ against fracture

Load case data: axial load $P = 30$ kips = 30,000 lbs

Find: cross section diameter $d$ in inches

Solution: make the cross-section cut shown in right figure. From FBD statics, the resultant internal force is $F = P$, aligned with the normal $\mathbf{n} // x$. Average normal stress is

$$\sigma_{\text{avg}} = \frac{F}{A} = \frac{P}{A} = \frac{P}{\pi d^2} = \frac{4 P}{\pi d^2}$$

Design condition $\sigma_{\text{avg}} \leq \sigma_{\text{fail}}/s_F$. Solving for $d$:

$$d \geq \sqrt{\frac{4 P s_F}{\pi \sigma_{\text{fail}}}} = \sqrt{\frac{30 \text{ kips} \times 8 \times 4}{\pi \times 40 \text{ kips/in}^2}} = 2.76 \text{ in}$$
Ex 2: Axially Loaded Ductile Bar

Material data: ductile alloy; fails by maximum shear at \( \tau_{\text{fail}} = 20 \text{ ksi} \)

Failure mode: yield by crystal slip at \( \pm 45^\circ \) from longitudinal bar \( x \) axis. Why \( 45^\circ \)?

This will be the subject of a problem in Recitation 1

Safety factor: \( s_F = 5 \) against yield

Load case data: axial load \( P = 30 \text{ kips} = 30,000 \text{ lbs} \) as in previous example

Find: cross section diameter \( d \) in inches

Solution: make the skew cut shown in right figure. From FBD statics, the resultant internal force is \( F = P \), aligned with \( +x \). Project \( F \) on cut plane to get tangential force \( F_t = F \cos 45^\circ = F/\sqrt{2} = P/\sqrt{2} \). The skew cut area, a.k.a. shear area, is \( A_s = A/\cos 45^\circ = A/\sqrt{2} \).

The average shear stress is

\[
\tau_{\text{avg}} = \frac{F_t}{A_s} = \frac{P/\sqrt{2}}{A/\sqrt{2}} = \frac{P}{\frac{1}{4}\pi \sqrt{2} d^2} = \frac{P}{\frac{1}{2}\pi d^2}
\]

Design condition \( \tau_{\text{avg}} \leq \tau_{\text{fail}}/s_F \)

Solving for \( d \):

\[
d \geq \sqrt{\frac{2 P s_F}{\pi \tau_{\text{fail}}}} = \sqrt{\frac{30 \text{ kips} \times 5 \times 2}{\pi \times 20 \text{ kips/in}^2}} = 2.19 \text{ in}
\]
Material data: Al alloy, may fail either by maximum normal tension $\sigma_{\text{fail}} = 268 \text{ MPa}$ or under maximum absolute shear $\tau_{\text{fail}} = 165 \text{ MPa}$

Failure mode: break and separation under $\sigma_{\text{fail}}$ at pin cross section; see right figure above

Safety factor: $s_F = 6$

Load case data: axial load $P = 5 \text{ kN} = 5,000 \text{ N}$ applied to pins

Geometric data: cross section thickness $t = 5 \text{ mm}$, pin diameter $d_{\text{pin}} = 12 \text{ mm}$

Find: cross section height $h$ in mm

Solution. Make a cut at pin cross section. Corrected area is $A_{\text{corr}} = A - A_{\text{pin}} = ht - d_{\text{pin}} t$.

Average normal stress over corrected area is $\sigma_{\text{avg}} = P/A_{\text{corr}} = P/((h - d_{\text{pin}})t)$.

Design condition: $\sigma_{\text{avg}} \leq \sigma_{\text{fail}}/s_F$. Solving for $h$ gives

$$h \geq \frac{P s_F}{\sigma_{\text{fail}} t} + d_{\text{pin}} = \frac{5,000 \text{ N} \times 6}{268 \text{ N/mm}^2 \times 5 \text{ mm}} + 12 \text{ mm} = 34.4 \text{ mm}$$
Ex 4: Pin-Connected Bar, Failure by Pin Shear Off

Material data: Al alloy, may fail either by maximum normal tension $\sigma_{fail} = 268$ MPa or under maximum absolute shear $\tau_{fail} = 165$ MPa

Failure mode: one pin shears off; see right figure above

Safety factor: $s_F = 6$

Load case data: axial load $P = 5$ kN = 5,000 N applied to pins

Geometric data: cross section thickness $t = 5$ mm, cross section height $h = 35$ mm and pin diameter $d_{pin} = 12$ mm (actually $h$ and $d_{pin}$ are not needed for this failure mode).

Find: design variable is distance $d_s$ in mm

Solution. Make two cuts as shown. Total shear area is $A_s = 2 \times d_s \times t$. Resultant shear force over shear area, from FBD, is $F_s = P/2 + P/2 = P$. Average shear stress: $\tau_{avg} = \frac{F_s}{A_s} = \frac{P}{2 \times d_s \times t}$. Design condition: $\tau_{avg} \leq \frac{\tau_{fail}}{s_F}$. Solving for $d_s$ gives

$$d_s \geq \frac{P \cdot s_F}{2 \times \tau_{fail} \cdot t} = \frac{5,000 \text{ N} \times 6}{2 \times 165 \text{ N/mm} \times 5 \text{ mm}} = 18.2 \text{ mm}$$
Ex 5: Bolt Connector Design Using Average Shear Stress

This example deals with the design of the bolt for connecting two truss members, as shown above, using the average shear stress approach.

Figures (a)–(d) display a set of FBDs to help understand the concept of a shear area through which the total axial force \( P \) is transmitted ("flows") from one member to the other; (c) displays the cut plane. The resultant tangential force on that plane is called \( F_s \) for shear force. From equilibrium \( F_s = P \). The shear area is marked red in (d).

It should be noted that the actual stress distribution in the vicinity of the shear area is quite complicated because of high stress gradients near contact zones as well as bending due to member centerline eccentricity (more about this later). To get a realistic picture one would need to carry out a time-consuming 3D finite element analysis. By assuming that in the place of the bolt section a uniformly distributed shear stress develops, however, a simple solution is readily found. Approximations involved in this assumption are supposed to be covered by the safety factor.
Ex 5: Bolt Connector Design (cont’d)

Figure from previous slide reproduced for convenience.

Material data: metal bolt (ductile); fails by maximum shear at $\tau_{\text{fail}}$

Failure mode: average shear stress over shear area, marked red in (d), reaches $\tau_{\text{fail}}$

Safety factor: $s_F$, typically 4 or more to compensate for neglecting stress concentrations as well as load transfer eccentricity

Load case data: axial load $P$ to be transmitted by bolt.

Find: bolt diameter $d_{\text{bolt}}$

Solution: make the cut shown in (c) and assume that shear stress over the shear area $A_s = A_{\text{bolt}} = \pi d_{\text{bolt}}^2 / 4$ is uniform, as discussed in the previous slide. That average is given by $\tau_{\text{avg}} = F_s / A_s = P / A_s = 4P / (\pi d_{\text{bolt}}^2)$. The design condition is $\tau_{\text{avg}} \leq \tau_{\text{fail}} / s_F$.

Solving for the bolt diameter gives

$$d_{\text{bolt}} \geq + \sqrt{\frac{4 \ P \ s_F}{\pi \ \tau_{\text{fail}}}}$$

It only remains to plug numbers.
Ex 6: Bolt Connector Design with Double Shear Area

One flaw of the connector studied in the foregoing example is load eccentricity caused by member centerline offset. The resulting moment may cause additional bending stresses as well as alignment problems. A better design that avoids that problem is shown above, in which configuration symmetry eliminates bending. To get the effective shear area, make two cuts as pictured in (c). As a consequence the effective shear area doubles to \( A_s = 2 \frac{\pi d_{bolt}^2}{2} \), and the average shear is \( \tau_{avg} = \frac{P}{A_s} = \frac{2P}{\pi d_{bolt}^2} \). With the same design criteria of the previous example, the bolt diameter is now given by

\[
d_{bolt} \geq \sqrt{\frac{2P s_F}{\pi \tau_{fail}}}\]

If \( P, s_F \) and \( \tau_{fail} \) remain the same, this \( d_{bolt} \) is about 30% smaller than for Example 5. Further \( s_F \) could be cut in view of the lack of bending. On the other hand, this kind of connection is likely to be costlier to fabricate. Next slide pictures a fabrication instance.
Double Shear Area Pinned Joint Fabrication Instance

This picture shows a high quality fabrication of a pinned joint with double shear area. Note that the two connected truss members have the same diameter. Axial forces are nicely centered. A typical use: well made car suspension.