Homework Exercises for Chapter 6— Solutions

EXERCISE 6.1

\[ P = - \int_V \rho g u_z dV. \]  \hspace{1cm} (E6.7)

EXERCISE 6.2

\[
\begin{align*}
\frac{\partial U}{\partial u_{x_1}} &= ak(L_x + \Delta_x) \\
\frac{\partial U}{\partial u_{y_1}} &= ak(L_y + \Delta_y) \\
\frac{\partial U}{\partial u_{x_2}} &= bk(L_x + \Delta_x) \\
\frac{\partial U}{\partial u_{y_2}} &= bk(L_y + \Delta_y)
\end{align*}
\]  \hspace{1cm} (E6.8)

where \( a = b = 1 - L/L_0 \). The extension to three dimensions is straightforward.

EXERCISE 6.3 (This solution needs to be improved, but the present one should do for now)

Let \( L_0 = L/\cos \alpha \) be the initial length of each spring and \( \delta \) the length change (which because of the assumed symmetry is the same for each spring) due to the crown displacement \( u \). Then from Figure E6.1,

\[
L_0 - \delta = \frac{L}{\cos \theta}, \quad \delta = L \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right) \]  \hspace{1cm} (E6.9)

The strain energy accumulated in the \( i^{th} \) spring \( (i = 1, 2) \) is \( U^{(i)} = \frac{1}{2} k \delta^2 \), and for the two springs

\[
U = U^{(1)} + U^{(2)} = kL^2 \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right)^2 \]  \hspace{1cm} (E6.10)

The work of the load \( P \) on the vertical displacement is evidently \( P = f u \); and so the total potential energy is

\[
\Pi = U - P = kL^2 \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right)^2 - fu \]  \hspace{1cm} (E6.11)

A dimensionless version of this equation results on dividing through by \( kL^2 \) and replacing \( \lambda = P/(kL) \) and \( \mu = u/(L \tan \alpha) \):

\[
\frac{\Pi}{kL^2} = \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right)^2 - \lambda \mu \tan \alpha \]  \hspace{1cm} (E6.12)

In the foregoing equations \( \theta \) and \( u \) (or \( \mu \)) are linked via the compatibility constraint

\[
\tan \alpha = \tan \theta + \frac{u}{L} \]  \hspace{1cm} (E6.13)

For further use we note that differentiating the above equation with respect to \( u \) or \( \mu \) we obtain

\[
\frac{\partial \theta}{\partial u} = -\frac{\cos^2 \theta}{L}, \quad \frac{\partial \theta}{\partial \mu} = -\cos^2 \theta \tan \alpha \]  \hspace{1cm} (E6.14)
(b) The exact equilibrium equation can be obtained by differentiating (E6.12)

\[ r = \frac{\partial \Pi}{\partial u} = \frac{\partial \Pi}{\partial \theta} \frac{\partial \theta}{\partial u} = -2kL^2 \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right) \frac{\sin \theta}{\cos^2 \theta} \frac{\partial \theta}{\partial u} - f = 0, \]  

(E6.15)

and substituting \( \partial \theta/\partial u \) from (E6.14),

\[ r = 2kL \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right) \sin \theta - f = 0. \]  

(E6.16)

This equation may also be directly derived by considering the equilibrium of forces along the vertical direction: \( f \) is balanced by \( 2k\delta \sin \theta \), with \( \delta \) given by the second of (E6.9).

In terms of dimensionless parameters we can write down three progressively more complex forms of (E6.16) upon dividing through by \( kL \):

\[ \frac{1}{kL} r(\lambda, \theta) = 2 \left( \frac{\sin \theta}{\cos \alpha} - \tan \theta \right) - \lambda = 0, \]  

(E6.17)

\[ \frac{1}{kL} r(\lambda, \theta, \mu) = 2 \tan \alpha \left( \frac{\sin \theta}{\sin \alpha} + \mu - 1 \right) - \lambda = 0, \]  

(E6.18)

\[ \frac{1}{kL} r(\lambda, \mu) = 2 \tan \alpha \left[ \frac{(1 - \mu)}{\sqrt{\cos^2 \alpha + [(1 - \mu) \sin \alpha]^2}} + \mu - 1 \right] - \lambda = 0. \]  

(E6.19)

(c) To locate limit points we may use the condition

\[ \frac{\partial \lambda}{\partial \mu} = \frac{\partial \lambda}{\partial \theta} \frac{\partial \theta}{\partial \mu} = 0. \]  

(E6.20)

From (E6.14) and (E6.17) we get

\[ 2 \left( \frac{\cos \theta_L}{\cos \alpha} - \frac{1}{\cos^2 \theta_L} \right) (-\cos^2 \theta_L \tan \alpha) = 0, \]  

(E6.21)

where \( \theta_L \) denotes the value of \( \theta \) at the limit points. Since the last factor does not vanish (unless \( \theta_L = 90^\circ \)), setting the first one to zero immediately yields the angular condition

\[ \cos^3 \theta_L = \cos \alpha, \]  

(E6.22)

and

\[ u_L = L(\tan \alpha - \tan \theta_L), \quad \mu_L = 1 - \frac{\tan \theta_L}{\tan \alpha}, \quad \lambda_L = 2 \left( \frac{\sin \theta_L}{\cos \alpha} - \tan \theta \right) \]  

(E6.23)

If \( \alpha = 30^\circ \) we get for the first limit load \( \cos \theta_{L1} = (\sqrt{3}/2)^{1/3} = 0.95318429 \); thus

\[ \theta_{L1} = 17.60120^\circ, \quad \mu_{L1} = 0.45052093, \quad \lambda_{L1} = 0.0638560 \]  

(E6.24)

(d) If the angles \( \alpha \) and \( \theta \) are fairly small we can insert the approximations

\[ \frac{1}{\cos \alpha} \approx \frac{1}{1 - \frac{1}{2} \alpha^2} \approx 1 + \frac{1}{2} \alpha^2, \quad \frac{1}{\cos \theta} \approx \frac{1}{1 - \frac{1}{2} \theta^2} \approx 1 + \frac{1}{2} \theta^2. \]  

(E6.25)
$\alpha = \pi/6; \quad (\ast \text{rise is 30 degrees} \ast)$

\[
\lambda_{\text{approx}} = \alpha^3 \mu (1-\mu) (2-\mu); \\
\lambda_{\text{exact}} = 2 \tan(\alpha) ((1-\mu)/\sqrt{\cos(\alpha)^2 + (1-\mu)^2 \sin(\alpha)^2}) + \mu - 1;
\]

Plot[{$\lambda_{\text{exact}}$, $\lambda_{\text{approx}}$}, {$\mu$, 0, 2.5}, PlotRange -> {-.1, .1}, PlotPoints -> 30];

Figure E6.3. Plot of $\lambda$ versus $\mu$ responses for Exercise E6.3, along with the Mathematica script that produced it.

into the potential energy (E6.11) to arrive at the quartic polynomial

$$\Pi \approx \frac{1}{4} k L^2 (\alpha^2 - \theta^2)^2 - fu. \quad (E6.26)$$

Similarly, replacing $\tan \alpha \approx \alpha$ and $\tan \theta \approx \theta$ into (E6.14) the relations that link $\mu$ and $\theta$ simplify to

$$\alpha \approx \frac{u}{L} + \theta, \quad \alpha \approx \alpha \mu + \theta, \quad \frac{\partial \theta}{\partial \mu} \approx -\frac{1}{L}, \quad \frac{\partial \theta}{\partial \mu} \approx -\alpha. \quad (E6.27)$$

The equilibrium equations reduce to cubic polynomials, three forms of which are

$$k L \theta (\alpha^2 - \theta^2) - p = 0, \quad (E6.28)$$

$$\theta(\alpha^2 - \theta^2) - \lambda = 0, \quad (E6.29)$$

$$\alpha (1-\mu) [\alpha^2 - \alpha^2 (1-\mu)^2] - \lambda = \alpha^3 \mu (1-\mu) (2-\mu) - \lambda = 0. \quad (E6.30)$$

The last form, (E6.30), shows that $\lambda$ vanishes at $\mu = 0, 1, 2$, as is physically obvious.

The equation that determines the limit points is

$$\frac{\partial \lambda}{\partial \theta} = 3 \theta_L - \alpha^2 = 0, \quad 3 \theta_L^2 = \alpha^2. \quad (E6.31)$$

For $\alpha = 30^\circ = \pi/6$ we get

$$\theta_L = \pm \frac{\pi}{6 \sqrt{3}} = \pm 0.30230 = \pm 17.32051^\circ, \quad \mu_{L1} = 2 - \mu_{L2} = 0.4226495, \quad \lambda_L = \pm 0.0552515. \quad (E6.32)$$

6–19
Chapter 6: CONSERVATIVE SYSTEMS

\[
r = \alpha^3 \mu^*(1-\mu) * (2-\mu) - \lambda; \quad q = D[r, \lambda]; \quad K = \text{Simplify}[D[r, \mu]]; \quad v = q/K; \quad \kappa = \text{Simplify}[(q*v)/(v*v)];
\]
\[
\kappa_0 = \kappa/.\mu \to 0; \quad \kappa = \text{Simplify}[(\kappa/\kappa_0)];
\]
\[\text{Print["q=",q," K=",K," v=",v," \kappa=",\kappa];}\]
\[\text{Plot[}[\kappa, \{\mu, 0, 0.4226495\}, \text{PlotStyle->AbsoluteThickness[2]}, \text{Frame->True}, \text{ImageSize->450}, \text{PlotRange->All}];\]

\[
q = 1 \quad K = \alpha^3(2-6\mu+3\mu^2) \quad \text{v} = \frac{1}{\alpha^3(2-6\mu+3\mu^2)} \quad \text{normalized } \kappa = 1 - 3\mu + \frac{3\mu^2}{2}
\]

**Figure E6.4.** Mathematica script for deriving the expression of the current stiffness parameter \(\kappa\) (also known as limit point sensor) for Exercise E6.4, and to plot it versus the dimensionless state \(\mu\). Printed results shown in lower (output) box.

---

The errors with respect to the exact solution (E6.24) are 1.7\% in \(\theta_L\) and 13.5\% in \(\lambda_L\); the latter being on the safe side.

(e) The plot of the approximate response curve (E6.30), \(\lambda = \alpha^3\mu(\mu - 1)(\mu - 2)\) for \(\alpha = \pi/6\) and of the exact response were produced Mathematica program listed on Figure E6.3 and “massaged” through Adobe Illustrator to place labels, etc.

**Exercise 6.4** Using the approximate residual given in (E6.30), the Mathematica script shown in the upper box of Figure E6.4 gives, after a minor rearrangement:

\[
\kappa = \frac{1}{2} \left( 3(1 - \mu)^2 - 1 \right). \quad (E6.33)
\]

Here \(\kappa\) has been normalized by dividing through \(\kappa(0)\). This \(\kappa\) goes from \(\kappa = 1\) at \(\mu = 0\) to \(\kappa = 0\) at \(\mu = \mu_{L1} \approx 0.4226495\). (If continued beyond \(\mu_{L1}\), \(\kappa\) would again vanish at \(\mu = \mu_{L2}\).) The expression in terms of \(\lambda\) would be very messy because \(\lambda\) is a cubic function of \(\mu\).

The plot of \(\kappa\) versus \(\theta\) produced through Mathematica and “massaged” through Adobe Illustrator is shown on Figure E6.5.