Chapter 4: ONE-PARAMETER RESIDUAL EQUATIONS

Homework Exercises for Chapter 4
One-Parameter Residual Equations – Solutions

EXERCISE 4.1
(a) The residual is separable because
\[ p = \begin{bmatrix} u_1 + 3u_2^2 \\ u_2 + 6u_1u_2 \end{bmatrix}, \quad f = \begin{bmatrix} 2\Lambda_1 \\ \Lambda_2 \end{bmatrix}. \] (E4.3)

(b) The internal and external forces are derivable from the potentials
\[ U = \frac{1}{2}(u_1^2 + u_2^2) + 3u_1u_2, \quad P = 2u_1\Lambda_1 + u_2\Lambda_2. \] (E4.4)

EXERCISE 4.2 First stage:
\[ f = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = (1 - \lambda) \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad q = \frac{\partial f}{\partial \lambda} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \] (E4.5)

Second stage:
\[ f = \begin{bmatrix} 2\Lambda_1 \\ \Lambda_2 \end{bmatrix} = (1 - \lambda) \begin{bmatrix} 0 \\ 5 \end{bmatrix} + 2\lambda \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 20\lambda \\ 5 + 5\lambda \end{bmatrix}, \quad q = \frac{\partial f}{\partial \lambda} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}. \] (E4.6)

Since each \( q \) is constant, the loading is proportional in each stage.

EXERCISE 4.3
(a) Yes; the only difference is that \( p_1 = 2\Lambda_1^2 \).
(b) For stage 1 \( q \) is the same as above. But in stage 2,
\[ q = \frac{\partial f}{\partial \lambda} = \begin{bmatrix} \frac{\partial(200\lambda^2)}{\partial \lambda} \\ \frac{\partial(5 + 5\lambda)}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 400\lambda \\ 5 \end{bmatrix}. \] (E4.7)

Therefore the loading is not proportional during the second stage.

EXERCISE 4.4 The tangent stiffness \( K \) and the incremental load vector \( q \) obtained from (E4.1) using equations (3.8) and (4.4), respectively are
\[ K = \begin{bmatrix} 1 & 6u_2 \\ 6u_2 & 1 + 6u_1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \] (E4.8)

The incremental velocity using (4.10) is
\[ v = K^{-1}q = \begin{bmatrix} 1 \\ 6u_2 \\ 1 + 6u_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{1}{6u_1 - 36u_2^2 + 1} \begin{bmatrix} -30u_2 \\ 5 \\ 1 \end{bmatrix}. \] (E4.9)

The tangent vectors are:
\[ t = \begin{bmatrix} v \\ 1 \end{bmatrix} = \begin{bmatrix} -30u_2 \\ 5 \\ 6u_2 - 36u_2^2 + 1 \\ 1 \end{bmatrix} \quad \text{and} \quad t \_b o l d = \frac{1}{f} v. \] (E4.10)
where $f$ is determined from (4.26) as

$$f = \left[ \frac{(1 + 6u_1 - 36u_2^2)^2 + 25 + 900u_2^2}{(1 + 6u_1 - 36u_2^2)^2} \right]^\frac{1}{2}. \quad (E4.11)$$

The hyperplane equation is

$$\frac{1}{6u_1 - 36u_2^2 + 1} [-30u_2(u_1 - u_{1p}) + 5(u_2 - u_{2p})] + \lambda - \lambda_p = 0. \quad (E4.12)$$

The differential equation of the orthogonal flow given by equation (4.22) is

$$\frac{1}{6u_1 - 36u_2^2 + 1} [-30u_2\dot{u}_1 + 5\dot{u}_2] + \dot{\lambda} = 0. \quad (E4.13)$$

**EXERCISE 4.5** $||r||$ is the distance to $r = 0$. Thus $||r|| = C$ are hypersurfaces of equal distance to an equilibrium path.

**EXERCISE 4.6** Yes. The most general expression of a curve in $(u, \lambda)$ space is the parametric form $u = u(t)$ and $\lambda = \lambda(t)$. The tangent direction is defined by $du/dt = \dot{u}$ and $d\lambda/dt = \dot{\lambda}$. Equation (4.17) is a special form obtained by taking $t \equiv \lambda$. 

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