CR Matrices
For Triangular
Shell Elements
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§B.1. Introduction

The element types tested in [332] are shown in Figure B.1. Each element has 6 degrees of freedom (DOF) per node: three translations and three rotations. These include shell as well as edge-attachable beam elements. Shells include the drilling DOF.

The linear internal force and stiffness matrix of the shell elements are constructed with the Assumed Natural Deviatoric Strain (ANDES) formulation [219,236,331,475,220,238,475,652] in terms of the deformational displacements and rotations. ANDES is a direct descendent of the Assumed Natural Strain (ANS) formulation of Park and Stanley (537) and the Free Formulation of Bergan and coworkers [79,81,85,86,88,210,495,496]. The derivation of the models is outlined in [332].

The matrices required to implement the EICR are $T$, $P$, $S$, $G$, $H$ and $L$. None of these depend on how the internal element force $\bar{p}^e$ and stiffness matrix $\bar{K}^e$ of the small-strain linear element are formed. In terms of implementation the EICR matrices can be classified into two groups:

(i) $T$, $H$ and $L$, as well as the T-projector component $P_u$ of $P$, are block diagonal matrices built up with $3 \times 3$ node blocks. These blocks can be formed by standard modules which are independent of the element type, as long as the element has the standard 6 DOFs per node. The only difference is the number of nodes.

(ii) The R-projector component of $P$, which is $P_\omega = SG$, does depend on element type, geometry and choice of CR frame through matrix $G$. These must be recoded for every change in those attributes.

In this Appendix we give the $G$ and $S$ matrices that appear in the “front end” of the EICR for the triangular shell element, as that illustrates the effect of CR frame selection. Matrices for the beam and quadrilateral element are given in [332].

In the following sections, element axes labels are changed from $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ to $\{\bar{x}, \bar{y}, \bar{z}\}$ to unclutter nodal subscripting. Likewise the displacement components $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ are relabeled $\{\bar{u}_x, \bar{u}_y, \bar{u}_z\}$.

**Figure B.1.** Elements tested in Part II. EICR matrices for the triangular shell elements are provided here.

§B.2. Spin-Lever Matrix

This $18 \times 3$ spin lever matrix is given by (written in transposed form to save space)

$$\vec{S} = \begin{bmatrix} \text{Spin}(\bar{x}_1) & -\text{Spin}(\bar{x}_2) & -\text{Spin}(\bar{x}_3) & I \end{bmatrix}^T$$

where $I$ is the $3 \times 3$ identity matrix, and $\bar{x}_a = [\bar{x}_a, \bar{y}_a, \bar{z}_a]^T$, the position vector of node $a$ in the deformed (current) configuration, measured in the element CR frame.
§B.3. Spin-Fitter Matrix

This $3 \times 18$ matrix, called $G$, connects the variation in rigid element spin to the incremental translations and spins at the nodes, both with respect to the CR frame. $G$ decomposes into three $3 \times 6$ submatrices, one for each node:

$$\delta \omega_r = \tilde{G} \delta \tilde{v}, \quad G = [ G_1 \ G_2 \ G_3 ], \quad \delta \tilde{v} = \begin{bmatrix} \delta \tilde{v}_1 \\ \delta \tilde{v}_2 \\ \delta \tilde{v}_3 \end{bmatrix}, \quad \delta \tilde{v}_a = \begin{bmatrix} \delta \tilde{u}_a \\ \delta \hat{\omega}_a \end{bmatrix}, \quad a = 1, 2, 3. \quad (B.2)$$

Submatrices $G_a$ depend on how the element CR frame is chosen. The origin of the frame is always placed at the element centroid. But various methods have been used to orient the CR axes $\{ \tilde{x}_i \}$. Three methods of historical or practical importance are described.

§B.3.1. $\tilde{G}$ by Side Alignment

This procedure is similar to that used by Rankin and coworkers [493,590,592]. They select side $1–3$ for $\tilde{x}_2$ and node 1 as frame origin. The approach used here aligns $\tilde{x}_1$ with side $1–2$ and picks the centroid as origin. Denote the cyclic permutations $i = 1, 2, 3$ by $j = 2, 3, 1$ and $k = 3, 1, 2$.

Then

$$\tilde{G}_1 = c_G \begin{bmatrix} 0 & 0 & \tilde{x}_{32} & 0 & 0 & 0 \\ 0 & 0 & \tilde{y}_{32} & 0 & 0 & 0 \\ 0 & -h_3 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{G}_2 = c_G \begin{bmatrix} 0 & 0 & \tilde{x}_{13} & 0 & 0 & 0 \\ 0 & 0 & \tilde{y}_{13} & 0 & 0 & 0 \\ 0 & h_3 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{G}_3 = c_G \begin{bmatrix} 0 & 0 & \tilde{x}_{32} & 0 & 0 & 0 \\ 0 & 0 & \tilde{y}_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (B.3)$$

in which $c_G = 1/(2A), h_i = 2A/L_i$ is the distance of corner $i$ to the opposite side, $A$ the triangle area, and $L_i$ the length of side $jk$, that is, that opposite corner $i$. This choice satisfies the decomposition property (?) that guards against unbalanced force effects while iterating for equilibrium. On the other hand it violates invariance: the choice of CR frame depends on node numbering, and different results may be obtained if the mesh is renumbered.

§B.3.2. $G$ by Least Square Angular Fit

Bjærum [92] and Nygård [495] place $CR$ in the plane of the deformed element with origin at node 1. The inplane orientation of the CR element is determined by a least square fit of the side angular errors. Referring to Figure B.2, the squared error is $d^2 = \phi_1^2 + \phi_2^2 + \phi_3^2$. Rotating by an additional angle $\chi$ this becomes $d^2(\chi) = (\phi_1 + \chi)^2 + (\phi_2 + \chi)^2 + (\phi + \chi)^2$. Minimization respect to $\chi$: $\partial d^2/\partial \chi = 0$ yields $\chi = -(\phi_1 + \phi_2 + \phi_3)/3$. Consequently the optimal in-plane position according to this criterion is given by the mean of the side angular errors.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figB2}
\caption{Side angular error measure for triangular shell element.}
\end{figure}
This condition yields for the nodal submatrices

\[
\tilde{G}_i = \frac{1}{2A} \begin{bmatrix}
0 & 0 & \tilde{x}_{kj} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{y}_{kj} & 0 & 0 \\
\frac{2A}{3} (\frac{s_{xy}}{L_j} - \frac{s_{xy}}{L_k}) & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

(B.4)

where \(L_i\) is the side length opposite corner \(i\) and \(\{s_{kx}, s_{ky}\}\) the projections of that side on the \(\{x, y\}\) axes. An advantage of this fitting method is that it satisfies invariance with respect to node numbering. On the other hand, the rotator gradient matrix cannot be decomposed as in (\?) leading to an approximate projector away from equilibrium. This disadvantage is not serious for a flat triangular element, however, since the CR and deformed configurations remain close on the assumption of small membrane strains.

A more serious shortcoming is that the procedure reintroduces the problem of spurious normal-to-the-plane rotations when an element with drilling freedoms is subjected to pure stretch. The difficulty is illustrated in Figure B.3, where a two triangle patch is subject to uniform stretch in the \(\tilde{y}\) direction. Under this state all rotations should vanish. The elements do rotate, however, because of the in-plane skewing of the diagonal. A deformatonal rotation is picked up since a predictor step gives no drilling rotations at the nodes, whereas the deformatonal drilling rotation is the total minus the rigid body rotation: \(\theta_d = \theta - \theta_r\).

![Figure B.3](image)

The problem is analogous to that discussed by Irons and Ahmad [384, p. 289] when defining node drilling freedoms as the mean of rotations of element sides meeting at that node; such elements grossly violate the patch test. This difficulty was overcome by Bergan and Felippa [86] by defining the node drilling freedom as the continuum mechanics rotation \(\theta_z = \frac{1}{2}(\partial \tilde{u}_y/\partial \tilde{x} - \partial \tilde{u}_x/\partial \tilde{y})\) at the node. It is seen that the problem of spurious drilling rotations has been reintroduced for the nonlinear case by the choice of CR frame positioning. In fact this problem becomes even more serious with the side alignment procedure described in the foregoing subsection.
§B.3.3. Fit According to CST Rotation

The infinitesimal drilling rotation of a plane stress CST element (also known as Turner triangle and linear triangle), is given by [219,236]:

$$\bar{\theta}_{\text{lin CST}} = \frac{1}{2} \left( \frac{\partial \bar{u}_y}{\partial \bar{x}} - \frac{\partial \bar{u}_x}{\partial \bar{y}} \right) = \frac{1}{4A} \left( \bar{x}_{23} \bar{u}_{x1} + \bar{x}_{31} \bar{u}_{x2} + \bar{x}_{12} \bar{u}_{x3} + \bar{y}_{23} \bar{u}_{y1} + \bar{y}_{31} \bar{u}_{y2} + \bar{y}_{12} \bar{u}_{y3} \right).$$  \hfill (B.5)

in which $\bar{x}_{ij} = \bar{x}_i - \bar{x}_j$, $\bar{y}_{ij} = \bar{x}_i - \bar{y}_j$, etc. The extension of this result to finite rotations can be achieved through a mean finite strain rotation introduced by Novozhilov [494, p. 31]; cf. also [721, Sec. 36]:

$$\tan \bar{\theta}_{\text{CST}} = \frac{\bar{\theta}_{\text{lin CST}}}{\sqrt{(1 + \bar{\epsilon}_{xx})(1 + \bar{\epsilon}_{yy}) - \frac{1}{4} \gamma_{xy}^2}}$$  \hfill (B.6)

in which $\bar{\epsilon}_{xx} = \partial \bar{u}_x/\partial \bar{x}$, $\bar{\epsilon}_{yy} = \partial \bar{u}_y/\partial \bar{y}$ and $\bar{\gamma}_{xy} = \partial \bar{u}_y/\partial \bar{x} + \partial \bar{u}_x/\partial \bar{y}$ are the infinitesimal strains (constant over the triangle) computed from the CST displacements. Novozhilov proves that this rotation measure is invariant with respect to the choice of CR axes $\{\bar{x}, \bar{y}\}$ since it is obtained as a rotational mean taken over a $2\pi$ sweep about $\bar{z}$. If this result is applied to the finite stretch path test of Figure B.3 it is found that the CR frames of both elements do not rotate, and the test is passed. The rotation gradient submatrices are

$$\tilde{G}_i = \frac{1}{2A} \begin{bmatrix} 0 & 0 & \bar{x}_{kj} & 0 & 0 & 0 \\ 0 & 0 & \bar{y}_{kj} & 0 & 0 & 0 \\ -\frac{1}{2} \bar{x}_{kj} & -\frac{1}{2} \bar{y}_{kj} & 0 & 0 & 0 & 0 \end{bmatrix}$$  \hfill (B.7)

in which $j, k$ denote cyclic permutations of $i = 1, 2, 3$. The resulting $G$ matrix satisfies the geometric separability condition.

§B.3.4. Best Fit by Minimum LS Deformation

The best fit solution by least-squares minimization of relative displacements given in Appendix C as equation (C.8) was obtained in 2000 [229]. Unlike the previous ones it has not been tested as part of a CR shell program. The measure is invariant, but it is not presently known whether it passes the stretch patch test, or if the associated $G$ satisfies the geometric separability condition (?).