Mass Templates in a Variational Framework
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A question that may be interest to FEM theoreticians: can any mass template be produced by a conventional variational framework? By “conventional” is meant based on shape functions injected in the kinetic energy. More precisely: velocities are interpolated over the element from nodal velocities using velocity shape functions (VSF), and the element kinetic energy \( T^e \) evaluated by integration. The mass matrix follows as the Hessian of \( T^e \) with respect to nodal velocities, as per (16.5). In short, a variationally derived mass matrix (VDMM). For practical template construction and customization, the variational interpretation is superfluous, since templates can be expediently postulated and algebraically customized. The reformulation may be worthwhile, however, for mathematical investigations, as well as linkage to work conducted by other researchers.

Presently it is unknown whether the template-to-VDMM connection for arbitrary elements can be established. It has been only investigated for the two simplest bar elements: Bar2 and Bar3. In both cases, the general template was considered. The findings may be summarized as follows:

1. VSF that reproduce the general template as a VDMM can be found. They are not unique.
2. For any template instance that deviates from the CMM, the VSF do not coincide with the displacement shape functions (DSF) used in the derivation of the element stiffness.
3. VSF that deviate from the DSF are noninterpolatory and nonconforming with respect to nodal velocities computed from the displacements by time differentiation. They do not necessarily satisfy the unit-sum condition (also called partition of unity in the literature). A uniform velocity field, however, must produce the exact kinetic energy.

Two simple elements are analyzed below.

§V.1. Variationally Derived Bar2 Mass Template

We investigate whether the general one-parameter Bar2 mass template (22.3) can be produced as a VDMM. The velocity field derived from the axial displacement \( u^e(x, t) \) is \( \dot{u}^e(x, t) = d^e(x, t)/dt \). Evaluation at the nodes yields the nodal velocities \( \dot{u}_1 \) and \( \dot{u}_2 \), collected in \( \dot{u}^e = [\dot{u}_1 \; \dot{u}_2]^T \). Let \( N_1(\xi) = (1 - \xi)/2 \) and \( N_2(\xi) = (1 + \xi)/2 \) denote the well known displacement shape functions (DS)F of Bar2, \( \xi \) being the usual iso-P natural coordinate. The element velocity interpolation is taken to be

\[
\dot{u}^e(\xi) = \dot{u}_1 \; N_{v1}(\xi) + \dot{u}_2 \; N_{v2}(\xi),
\]

in which the velocity shape functions (VSF) \( N_{v1} \) and \( N_{v2} \) are linked to the DSF through the linear map

\[
N_{v1}(\xi) = (1 + \frac{1}{2} \delta_1) \; N_1(\xi) + \frac{1}{2} \delta_2 \; N_2(\xi), \quad N_{v2}(\xi) = \frac{1}{2} \delta_2 \; N_2(\xi) + (1 + \frac{1}{2} \delta_1) \; N_1(\xi). \tag{V.2}
\]

In (V.2), \( \delta_1 \) and \( \delta_2 \) are functions of the template parameter (but not of \( \xi \)), representing the deviations of the VSF from the DSF. Note that prismatic bar symmetry is built-in: \( N_{v1}(\xi) = N_{v2}(\xi) \). The associated kinetic energy \( T^e \) is \( \rho \; A \; (\ell/2) \int^1_{-1} (\dot{u}^e(\xi))^2 \; d\xi \), which can be evaluated either analytically or through 2-point Gauss integration. Taking its Hessian with respect to \( \dot{u}^e \) gives a mass matrix denoted by \( M^e_\delta \) below. As for the Bar2 template, it is preferable to use the alternative form \( M^e_\chi \) of (22.4) rather than \( M^e_\mu \) of (22.3) because solutions are simpler. Summarizing, the two matrices to
Next we find whether the general mass template for Bar3 can be derived variationally. The well known displacement shape functions are \( N_1(\xi) = \xi(\xi - 1)/2 \), \( N_2(\xi) = \xi(\xi + 1)/2 \), and \( N_3(\xi) = \xi/2 \).
In terms of the $\delta$, Here to be evaluated either analytically or from 3-point Gauss integration. Taking its Hessian with respect to $\delta$, matrix properties can be discerned visually: therein. Except for the CMM they depart from the DSF, and are nonconforming. Some mass matrices are produced. The VSF produced by (V.8) are plotted in Figure V.2 for nine Bar3 mass instances, as labeled against (23.2). Matching entries gives 8 solutions, of which the one that yields $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$ for the CMM ($\chi_1 = 5/2$, $\chi_2 = 3/2$, $\chi_3 = 2/3$) is picked:

$$\delta_1 = \phi_1 - 1/2, \quad \delta_2 = \phi_2 - 1/2, \quad \delta_3 = 3\phi_3/2 - 1, \quad \delta_4 = 1 - 2\phi_2 + \phi_3 - 5\phi_4/2,$$

in which $\phi_1 = \sqrt{\chi_2/10}$, $\phi_2 = \sqrt{\chi_1/6}$, $\phi_3 = \sqrt{\chi_3/\chi_1}$, and $\phi_4 = \sqrt{5(1 - \chi_3/\chi_1)}$. Except for the CMM, these VSF do not verify the strong (pointwise) unit sum condition $N_{v1} + N_{v2} + N_{v3} = 1$ for each $\xi$, but do satisfy the more lenient element mass conservation constraint

$$\frac{1}{2} \int_{-1}^{1} (N_{v1} + N_{v2} + N_{v3})^2 d\xi = 1.$$  

In terms of the $\delta$, (V.9) is $12\delta_1^2 + 15\delta_2^2 + 10\delta_3(3 + \delta_4) + \delta_4(10 + 3\delta_4) + 4\delta_5(5 + 5\delta_3 + 3\delta_4) = 0$. The VSF produced by (V.8) are plotted in Figure V.2 for nine Bar3 mass instances, as labeled therein. Except for the CMM they depart from the DSF, and are nonconforming. Some mass matrix properties can be discerned visually:

- For the diagonally lumped instances SLMM and BLFD shown in Figure V.2(b,e), two VSF vanish at each of the sample points $\xi \in \{0, \pm \sqrt{3/5}\}$ of the 3-point Gauss rule. Those points are marked in the Figure. This feature effectively energy-orthogonalizes the VSF in the sense of kinetic energy, since all cross integrals $\int_{-1}^{1} N_{vi} N_{vj} d\xi$ for $i \neq j$ vanish. As a result, diagonal mass matrices are produced.

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Figure V.2. Velocity shape functions (VSF) that produce the general Bar3 mass template (23.7) in a variational framework, for the nine labeled instances.

- The VSF for the singular mass instance BSSM shown in Figure V.2(f), clearly displays linear dependence among the VSF.

Aside from those special cases, it is difficult to draw general conclusions from a glance at Figure V.2 as to performance. For example, why does the VSF in (d) provide the best low frequency matching? Shapes for say, (a) through (e) look quite similar (once you’ve seen one parabola ...). The obvious conclusion: Fourier analysis is a much sharper tool in dynamics.

§V.3. A Comment on the Variational Formulations of Elastodynamics

The use of VSF that differ from DSF dates back to the early days of FEM. It was done, for example, in [203] for the HCT plate bending element, following suggestions by R. W. Clough. (The consistent mass of that tricubic macroelement was quite complicated for hand derivations in 1966.) The idea can be incorporated into the well-known stationary-action variational principle (VP) of elastodynamics, called Hamilton-Kirchhoff by Gurtin [321, p. 225], by weakening the temporal kinematic link.
That minor generalization of the primal VP of elastodynamics should not be confused with the use of dual (also called complementary or reciprocal) forms. Research in that subject took off with Toupin’s formulation [748] of a dual form of Hamilton’s principle for a system of mass particles with interaction impulses as unknown variables. For corrections and evolution into continua see [196,718] and references therein. FEM applications to vibrations and dynamics emerged during the early 1970s; see e.g., [277,300,717], but have stagnated since. Reason: impulse DOF are foreign to the DSM, which dominates general purpose codes.