Shape Function Magic
'Magic' Means *Direct*  
("*by inspection*")

Do in 15 minutes what took smart people several months  
(and less gifted, several years)

But ... it looks like magic to the uninitiated
Shape Function Requirements

(A) Interpolation
(B) Local Support
(C) Continuity (Intra- & Inter-Element)
(D) Completeness

See Sec 18.1 for more detailed statement of (A) through (D). Implications of the last two requirements as regards convergence are discussed in Chapter 19.
**Direct Construction of Shape Functions:**
*Are Conditions Automatically Satisfied?*

(A) *Interpolation*  
Yes: by construction except scale factor

(B) *Local Support*  
Often yes, but not always possible

(C) *Continuity*  
No: *a posteriori* check necessary

(D) *Completeness*  
Satisfied if (B,C) are met and the sum of shape functions is identically one.  
Section 16.6 of Notes (advanced material) provides details
Direct Construction of Shape Functions
as "Line Products"

\[
N^e_i \overset{\text{guess}}{=} c_i L_1 L_2 \ldots L_m
\]

where \( L_k = 0 \) are equations of "lines" expressed in natural coordinates, that cross all nodes except \( i \)
The Three Node Linear Triangle

At node 1, $N_1^e = 1$ whence $c_1 = 1$
and $N_1^e = \zeta_1$. Likewise for $N_2^e$ and $N_3^e$. 

$$N_1^e \text{ guess} = c_1 L_1 = c_1 L_{2,3}$$
Three Node Triangle Shape Function Plot

\[ N_1 = \zeta_1 \]
The Six Node Triangle - Corner Node

\[ N^e_{\text{guess}} = c_1 L_{2-5-3} L_{4-6} = c_1 \zeta_1 (\zeta_1 - 1/2) \]

For rest of derivation, see Notes
The Six Node Triangle - Midside Node

\[ N_4^e \text{ guess} = c_4 \ L_{2-5-3} \ L_{3-6-1} = c_4 \ \zeta_1 \ \zeta_2 \]

For rest of derivation, see Notes
The Six Node Triangle: Shape Function Plots

\[ N_1^e = \zeta_1(2\zeta_1 - 1) \]

\[ N_4^e = 4\zeta_1\zeta_2 \]
The Four Node Bilinear Quad

For rest of derivation, see Notes
The Four Node Bilinear Quad:
Shape Function Plot

\[ N_1^e = \frac{1}{4} (1 - \xi)(1 - \eta) \]
The Nine Node Biquadratic Quad
Corner Node Shape Function

\[ N_1^e \text{ guess} = c_1 L_{2-3} L_{3-4} L_{5-7} L_{6-8} = c_1 (\xi - 1)(\eta - 1)\xi\eta \]

For rest of derivation, see Notes
The Nine Node Biquadratic Quad
Internal Node Shape Function

\[ N_9^e = c_9 L_{1-2}L_{2-3}L_{3-4}L_{4-1} = c_9 (\xi - 1)(\eta - 1)(\xi + 1)(\eta + 1) \]

For rest of derivation, see Notes
The Nine-Node Biquadratic Quad: Shape Function Plots

\( N_1^e = \frac{1}{4}(\xi - 1)(\eta - 1)\xi\eta \)

\( N_5^e = \frac{1}{2}(1 - \xi^2)\eta(\eta - 1) \) (back view)

\( N_9^e = (1 - \xi^2)(1 - \eta^2) \)
The Eight-Node "Serendipity" Quad Corner Node Shape Function

\[ N_1^e = c_1 L_{2-3} L_{3-4} L_{5-8} = c_1 (\xi - 1)(\eta - 1)(1 + \xi + \eta) \]

For rest of derivation, see Notes
Can the Magic Wand Fail? Yes

Method also needs modifications in *transition elements*. One example is covered in the next two slides.

(Exercise 18.6)
Transition Element Example

For $N_1^e$ try the magic wand: product of side 2-3 ($\zeta_1 = 0$) and median 3-4 ($\zeta_1 = \zeta_2$):

$$N_1^e \text{ guess } = c_1 \zeta_1 (\zeta_1 - \zeta_2) \quad N_1^e(1, 0, 0) = 1 = c_1$$

No good: fails compatibility over side 1-2
Transition Element Example (cont'd)

Next, try the shape function of the linear 3-node triangle plus a correction:

\[ N_1^e \text{ guess} = \zeta_1 + c_1 \zeta_1 \zeta_2 \]

Coefficient \( c_1 \) is determined by requiring this shape function vanish at midside node 4:

\[ N_1^e \left( \frac{1}{2}, \frac{1}{2}, 0 \right) = \frac{1}{2} + c_1 \frac{1}{4} = 0, \]

whence \( c_1 = -2 \) and

\[ N_1^e = \zeta_1 - 2 \zeta_1 \zeta_2 \]

works