The Plane Stress Problem
Plate in Plane Stress

Inplane dimensions: in $x,y$ plane

Thickness dimension or transverse dimension

Top surface
Mathematical Idealization as a Two Dimensional Problem
Plane Stress Physical Assumptions

Plate is flat and has a symmetry plane (the midplane)

All loads and support conditions are midplane symmetric

Thickness dimension is much smaller than inplane dimensions

Inplane displacements, strains and stresses uniform through thickness

Transverse stresses $\sigma_{zz}$, $\sigma_{xz}$ and $\sigma_{yz}$ negligible, set to 0

Unessential but used in this course:

Plate fabricated of homogeneous material through thickness
Notation for Stresses, Strains, Forces, Displacements

Thin plate in plane stress

In-plane internal forces

\[
\begin{align*}
&\sigma_{xx}, \sigma_{yy}, \sigma_{xy} = \sigma_{yx} \\
&p_{xx}, p_{yy}, p_{xy}, p_{yx} \\
&\left(\begin{array}{c}
\frac{dy}{dx} \\
\frac{dx}{dy}
\end{array}\right)
\end{align*}
\]

In-plane stresses

\[
\begin{align*}
&\sigma_{xx}, \sigma_{yy}, \sigma_{xy} = \sigma_{yx} \\
\end{align*}
\]

In-plane body forces

\[
\begin{align*}
&b_x, b_y \\
&\left(\begin{array}{c}
\frac{dy}{dx} \\
\frac{dx}{dy}
\end{array}\right)
\end{align*}
\]

In-plane strains

\[
\begin{align*}
&e_{xx}, e_{yy}, e_{xy} = e_{yx} \\
\end{align*}
\]

In-plane displacements

\[
\begin{align*}
&u_x, u_y \\
&\left(\begin{array}{c}
\frac{dy}{dx} \\
\frac{dx}{dy}
\end{array}\right)
\end{align*}
\]
Inplane Forces are Obtained by Stress Integration Through Thickness

Inplane stresses

\[ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy} = \sigma_{yx} \]

Inplane internal forces
(also called membrane forces)
Plane Stress Boundary Conditions

Boundary displacements $\hat{\mathbf{u}}$ are prescribed on $\Gamma_u$ (figure depicts fixity condition).

Boundary tractions $\hat{\mathbf{t}}$ or boundary forces $\hat{\mathbf{q}}$ are prescribed on $\Gamma_t$.

Stress BC details (decomposition of forces $\hat{\mathbf{q}}$ would be similar).
The Plane Stress Problem

Given:

- geometry
- material properties
- wall fabrication (thickness only for homogeneous plates)
- applied body forces
- boundary conditions:
  - prescribed boundary forces or tractions
  - prescribed displacements

Find:

- inplane displacements
- inplane strains
- inplane stresses and/or internal forces
Matrix Notation for Internal Fields

\[ \mathbf{u}(x, y) = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} \quad \text{displacements} \]

\[ \mathbf{e}(x, y) = \begin{bmatrix} e_{xx}(x, y) \\ e_{yy}(x, y) \\ 2e_{xy}(x, y) \end{bmatrix} \quad \text{strains (factor of 2 in } e_{xy} \text{ simplifies "energy dot products")} \]

\[ \mathbf{\sigma}(x, y) = \begin{bmatrix} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ \sigma_{xy}(x, y) \end{bmatrix} \quad \text{stresses} \]
Governing Plane Stress Elasticity Equations in Matrix Form

\[
\begin{bmatrix}
  e_{xx} \\
  e_{yy} \\
  2e_{xy}
\end{bmatrix} = \begin{bmatrix}
  \frac{\partial}{\partial x} & 0 \\
  0 & \frac{\partial}{\partial y} \\
  \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} \begin{bmatrix}
  u_x \\
  u_y
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \sigma_{xx} \\
  \sigma_{yy} \\
  \sigma_{xy}
\end{bmatrix} = \begin{bmatrix}
  E_{11} & E_{12} & E_{13} \\
  E_{12} & E_{22} & E_{23} \\
  E_{13} & E_{23} & E_{33}
\end{bmatrix} \begin{bmatrix}
  e_{xx} \\
  e_{yy} \\
  2e_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\
  0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} \begin{bmatrix}
  \sigma_{xx} \\
  \sigma_{yy} \\
  \sigma_{xy}
\end{bmatrix} + \begin{bmatrix}
  b_x \\
  b_y
\end{bmatrix} = \begin{bmatrix}
  0
\end{bmatrix}
\]

or

\[
e = Du \quad \sigma = Ee \quad D^T \sigma + b = 0
\]
Introduction to FEM

Strong-Form Tonti Diagram of Plane Stress Governing Equations

- **Displacement BCs**: $u = \hat{u}$ on $\Gamma_u$
- **Kinematic**: $e = D u$ in $\Omega$
- **Constitutive**: \( \sigma = E e \) or \( e = C \sigma \) in $\Omega$
- **Equilibrium**: $D \sigma + b = 0$ in $\Omega$
- **Stresses**: $\sigma$
- **Force BCs**: $\sigma^T n = \hat{t}$ or $p^T n = \hat{q}$ on $\Gamma_t$
- **Prescribed tractions $t$ or forces $q$**

- **Displacements**: $u$
- **Body forces**: $b$
- **Strains**: $e$
- **Prescribed displacements $\hat{u}$**

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Introduction to FEM

TPE-Based Weak Form Diagram of Plane Stress Governing Equations

\[
\delta \Pi = 0 \quad \text{in} \quad \Omega
\]

\[
e = D u
\quad \text{in} \quad \Omega
\]

\[
u = u^\text{\dagger}\quad \text{on} \quad \Gamma
\]

\[
\sigma = E e
\quad \text{in} \quad \Omega
\]

Displacement BCs

Kinematic

Constitutive

Equilibrium (weak)

Force BCs (weak)

Prescribed tractions \( t \) or forces \( q \)

\( \delta \Pi = 0 \quad \text{on} \quad \Gamma_t \)
Total Potential Energy of Plate in Plane Stress

\[ \Pi = U - W \]

\[ U = \frac{1}{2} \int_{\Omega} h \sigma^T e \, d\Omega = \frac{1}{2} \int_{\Omega} h e^T E e \, d\Omega \]

\[ W = \int_{\Omega} h u^T b \, d\Omega + \int_{\Gamma_t} h u^T t \, d\Gamma \]

*body forces* \hspace{1cm} *boundary tractions*
Discretization into Plane Stress
Finite Elements
Plane Stress Element Geometries and Node Configurations

$n = 3$

$n = 4$

$n = 6$

$n = 12$
Total Potential Energy of Plane Stress Element

\[ \Pi^e \gamma = U^e - W^e \]

\[ U^e = \frac{1}{2} \int_{\Omega^e} h \sigma^T e \, d\Omega^e = \frac{1}{2} \int_{\Omega^e} h e^T E e \, d\Omega^e \]

\[ W^e = \int_{\Omega^e} h u^T b \, d\Omega^e + \int_{\Gamma^e} h u^T t \, d\Gamma^e \]
Constructing a Displacement Assumed Element

Node displacement vector

\[ \mathbf{u}^e = [u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn}]^T \]

Displacement interpolation over element

\[
\begin{bmatrix}
  u_x(x,y) \\
  u_y(x,y)
\end{bmatrix} =
\begin{bmatrix}
  N_1^e & 0 & N_2^e & 0 & \ldots & N_n^e & 0 \\
  0 & N_1^e & 0 & N_2^e & \ldots & 0 & N_n^e
\end{bmatrix}
\begin{bmatrix}
  \mathbf{u}^e
\end{bmatrix}
\]

\[ = \mathbf{N} \quad \mathbf{u}^e \]

\( \mathbf{N} \) is called the shape function matrix
It has order \( 2 \times 2n \)
Differentiate the displacement interpolation \( x, y \) to get the strain-displacement relation

\[
e(x, y) = \begin{bmatrix}
\frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \ldots & \frac{\partial N_n^e}{\partial x} & 0 \\
0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \ldots & 0 & \frac{\partial N_n^e}{\partial y} \\
\frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \ldots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x}
\end{bmatrix}
\]

\( u^e = B \) \( u^e \)

\( B \) is called the strain-displacement matrix

It has order \( 3 \times 2n \)
Element Construction (cont'd)

Element total potential energy

\[ \Pi^e = \frac{1}{2} \mathbf{u}^e \mathbf{K}^e \mathbf{u}^e - \mathbf{u}^e \mathbf{f}^e \]

Element stiffness matrix

\[ \mathbf{K}^e = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} \, d\Omega^e \]

Consistent node force vector

\[ \mathbf{f}^e = \int_{\Omega^e} h \mathbf{N}^T \mathbf{b} \, d\Omega^e + \int_{\Gamma^e} h \mathbf{N}^T \mathbf{t} \, d\Gamma^e \]

due to: body force    due to: surface tractions
Requirements on Finite Element Shape Functions

**Interpolation** Condition

\[ N_i \text{ takes on value 1 at node } i, \quad 0 \text{ at all other nodes} \]

**Continuity** (intra- and inter-element) and **Completeness** Conditions

are covered later in the course (Chs. 18-19)