Modeling Methods for Silent Boundaries in Infinite Media

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1 Introduction

In several problems of interest in computational modeling there is a media that is either infinite or semi-infinite. Some examples include: modeling the ocean when doing ship analysis, modeling the soil underneath a footing, and modeling the air around an airplane. However, every finite element model must be terminated at some finite boundary. Unfortunately, for problems involving wave propagation analysis, the usual finite boundary of the finite element model will cause the elastic waves to be reflected and superimposed with the progressing waves. Therefore, we wish to create a boundary, which is perfectly radiating to outgoing waves (no spurious reflections) and transparent to incoming waves. In addition, we would like the boundary to be as close to the finite structure as possible for computational efficiency.

A simple solution to the problem is to move the boundary a great distance away from the finite structure so that the boundary does not influence the results. However, this violates the concept of computational efficiency. Hence, we need an artificial boundary condition to simulate a model without any finite boundary. Through a literature research there appears to be two basic approaches to this boundary condition. The first is to create boundary elements with special properties to absorb the outgoing waves. The second approach is to create infinite elements. This report will discuss three concepts of the boundary element approach: plane wave approximation, viscous damping boundary method, and perfectly matched layers. A brief introduction into infinite elements will be presented as well.

2 Plane Wave Approximation

The plane wave approximation (PWA) is a boundary element method for creating a silent boundary. It is the lowest order member of the early-time approximation family. An early-time approximation approaches exactness as $t \to 0$ [1]. It is valid for short acoustic wavelengths (high frequencies) and is useful for modeling early stages of the transient interaction [3]. Derivations of the early-time approximations can be found in the paper by Felippa [1]. These early-time approximations do not treat added mass affects, which are valid for long acoustic wave lengths. To include the added mass affects one would want to use a doubly asymptotic approximation (DAA) [4]. The advantage of using an early-time approximation instead of the DAA is that it is less expensive in terms of memory storage and easier to implement.
Implementation of the PWA is fairly straightforward. It is a boundary element that is constructed on the finite extent of the infinite domain. For simplicity, it is best to construct the PWA elements so that the nodes coincide with the nodes of the media being modeled. An example of the use of the PWA is shown in figure (1). Here a dam and the associated fluid are modeled. Instead of modeling the fluid to the shoreline on the opposite end of the dam, we have used a PWA on the left side of the fluid. The system will experience an earthquake in the longitudinal direction. In order to evaluate the dam’s response accurately, we do not want reflecting waves from the boundary. Once again, for this model the PWA elements lie on the left most face of the brick elements of the fluid, and the PWA nodes coincide with the fluid nodes.

In order to prevent reflecting waves the PWA node velocities are modeled as

$$p = \rho c v,$$

where \( p \) is a vector of pressures corresponding to the degrees of freedom (DOF), \( \rho \) is the density of the media, \( c \) is the speed of sound in the media, and \( v \) are the DOF velocity components. Thus, the velocity of the PWA nodes are determined by the pressure acting at the nodes. From the above formulation we can easily see that the PWA acts a damper to the incoming pressure; thus, reducing the reflection of the pressure wave back towards the structure. Hence, interpretation in terms of mechanical components of the PWA fluid element is nothing more than a \( \rho c \) dashpot \([1]\). Thus, for modeling the interaction between the infinite media and the PWA, the infinite media sends pressures to the PWA and the PWA in turn send velocity (or displacement) values of the nodes back to the media.

For the example of the dam-fluid system under seismic excitation, pressure waves were examined as they encountered the PWA. During the video of this phenomenon one could easily see that the pressure waves were in fact absorbed by the PWA.
In summary, the plane wave approximation is an ideal silent boundary for plane waves propagating through a fluid media. Its implementation in terms of boundary elements is simple and straightforward. It works well for a staggered solution procedure. It is computationally inexpensive in comparison to the DAA silent boundary method. However, it does not treat added mass affects.

### 3 Viscous Damping Boundary Method

In the previous section we discussed a silent boundary that is useful for fluid media. Next, we examine a similar method, known as the viscous damping boundary method (VDB), that is useful for elastic infinite media. The VDB is similar to the plane wave approximation method in that the VDB uses the concept of applying viscous dampers to the DOF on the boundary element. However, for an elastic media, such as soil, there is primary waves and secondary waves that travel through the media. Thus, the speed of these waves is used instead of the speed of sound in the fluid, which was used for the PWA. Primary waves are compressional waves that travel through solids, liquids, and gases. Secondary waves are shear waves that only travel through solids (see figure 2 and figure 3).

In order to develop a boundary condition that ensures that all energy arriving at the boundary is absorbed, Lysmer and Kuhlemeyer [6] found that the most promising expression for the boundary condition is expressed by the following equations.

\[
\begin{align*}
\sigma_{zz} &= a \rho V_p \dot{u}_z & \text{Primary wave} \\
\tau_{xz} &= b \rho V_s \dot{u}_x & \text{Secondary wave} \\
\tau_{yz} &= b \rho V_s \dot{u}_y & \text{Secondary wave}
\end{align*}
\]

These equations are formulated for an incident wave consisting of a primary and secondary wave that act at an angle \( \theta \) from the z-axis as shown in figure (4) [2]. In the above equations, \( \rho \) is the density of the media, \( \dot{u}_x, \dot{u}_y, \dot{u}_z \) represent the velocities in the x,y,z-direction, \( V_p \) and \( V_s \) are the
velocities of the primary and secondary waves, and $a$, $b$ are dimensionless parameters. For small incident angles ($\theta < 30^\circ$), the most effective value for the dimensionless parameters ($a$ and $b$) is one [6]. The primary and secondary velocities can be computed as

$$V_p = \sqrt{\frac{(1-\nu)E}{(1-2\nu)(1+\nu)\rho}},$$
$$V_s = \sqrt{\frac{E}{2(1+\nu)\rho}},$$

where $E$ is the modulus of elasticity, and $\nu$ is the Poisson’s ratio.

Implementation of the method is fairly simple. It amounts to modifying the damping matrix in the standard finite element discretized equation of motion ($M\ddot{u} + C\dot{u} + Ku = f$). This intuitively makes sense, because in observation of the above equations we are adding nothing more than a $\rho V_p$ dashpot and $\rho V_s$ dashpot at the boundary DOF. To modify the damping matrix, one modifies the appropriate diagonal damping terms with the following equations

$$c_{ii} = \int_{\Gamma} a \rho V_p d\Gamma \quad \text{Primary wave, z-direction},$$
$$c_{ii} = \int_{\Gamma} b \rho V_s d\Gamma \quad \text{Secondary wave, x or y direction},$$

where $\Gamma$ is the surface area of the element.

An example of the effectiveness of this method is demonstrated in the following problem. A dam sits on top of sandstone as shown in figure (5). The sandstone is modeled to finite extent and then a VDB is applied to the left and right sides of the sandstone. A hydrostatic pressure is applied to the dam face and the displacement of the dam crest is monitored. This displacement of the dam crest from the model, as shown in figure (5), is compared to the dam crest displacements of a model where both sides of the sandstone is extended to a much greater extent. In both models a 5\% damping ratio is introduced into the sandstone. As can be seen in figure (6), the viscous damping method closely matches the extended mesh.
In summary, the viscous damping method is a simple silent boundary to implement and it is computational efficient, because the damping matrix is the only portion that is modified by this method. Because the absorption characteristics are independent of the frequency of the wave, the boundary can absorb both harmonic and nonharmonic waves [6]. However, the effectiveness of the absorption depends on the waves incident angle.

4 Perfectly Matched Layers

A newly discovered silent boundary is the perfectly matched layers method, first introduced by Berenger in 1994 for the use of electromagnetic waves. The concept is designed to absorb thoroughly any incident wave without reflection, for any incident angle and at any frequency before discretization. This method is growing rapidly and has been used in many fields, ranging from eddy-current problems to wave propagation in elastic media. The reason is that it is easy to formulate with a relatively small computational cost.

The main concept is to surround the computation domain at the infinite media boundary with a highly absorbing boundary layer, as shown in figure (7) [7]. Generally, the boundary layer is made of the same elements as the computational domain. The formulation of the matrices are the same method for both the computational domain and the boundary layer; however, the boundary layer as slightly different properties.

To determine the properties of the boundary layer, let us begin by examining the governing equations for the propagation of a linear elastic wave in any solid material. The first equation is the equation of motion

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \mathbf{T}, \quad (1)$$

where \( \rho \) is the media density, \( \mathbf{u} = (u_x, u_y, u_z)^T \) is the displacement vector, and \( \mathbf{T} \) is the stress tensor. The next governing equation is the constitutive equation

$$\mathbf{T} = T_{ij} = c_{ijkl} S_{kl} = \mathbf{C} : \mathbf{S}, \quad (2)$$
where \( S \) is the strain tensor, and \( C \) is the fourth order elasticity tensor, whose elements are \( c_{ijkl} (i,j,k,l = x,y,z) \). The strain tensor is related to the displacements by
\[
S_{k,l} = \frac{1}{2}(u_{k,l} + u_{l,k}),
\]
where the summation convention is applied, and \( u_{k,l} = \frac{\partial u_k}{\partial l} \). Due to the symmetry of the elasticity tensor of real solids, \( c_{ijkl} = c_{ijlk} \), we can rewrite the constitutive equation (2) in the form of
\[
T_{ij} = c_{ijkl}u_{l,k}, \quad (vector \ and \ tensor \ form) \ T = C : \nabla \mathbf{u}. \tag{3}
\]
Combining equations (1) and (3) we obtain the elastic wave equation,
\[
-\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot C : \nabla \mathbf{u}.
\]
In the frequency domain we can write the wave equation as
\[
\rho \omega^2 \mathbf{u} = \nabla \cdot C : \nabla \mathbf{u}.
\]
In a homogeneous, isotropic medium, this equation permits plane-wave solutions of the form

$$u(x, t) = Ae^{i(k \cdot x - \omega t)},$$

where $A$ represents the amplitude and polarization of the plane wave, $k$ is its wave vector in cartesian coordinates, and $x$ is the position vector [5]. The objective of the PML method is to construct a new wave equation that creates plane waves that decay exponentially in the PML in the direction of the PML. For instance, if we want the waves to decrease in the $x$-direction, then the new wave equation should have the form

$$u(x, t) = Ae^{i(k \cdot x - \omega t) - \zeta x},$$

where $\zeta$ is only provided for the PML region and produces the desired decay rate. Thus, we can see from the above equation that we want to introduce a new variable of the form

$$\tilde{x} = x - \zeta,$$

into the governing equations once the wave enters the PML region. This new variable is the complex coordinate variable.

This complex coordinate variable can also be introduced by the complex coordinate stretching method as done by Chew and Weedon in 1994, where

$$\tilde{x}_j = \int_0^{x_j} \gamma_j(\hat{x}) d\hat{x}, \quad (j = 1, 2, 3),$$

$$\lambda_j = 1 - i\gamma$$

Here $\gamma$ is the attenuation in the PML region. This $\gamma$ will be zero in the computational domain. Thus,

$$\zeta = \int_0^{x_j} \gamma(\hat{x}) d\hat{x},$$

to produce

$$\tilde{x} = x - \zeta.$$

The complex coordinate stretching function, $\lambda$ in equation (4), is continuous. Due to this continuity, we can notice that

$$\frac{\partial \tilde{x}_j}{\partial x_j} = \lambda_j(x_j) \implies \frac{\partial}{\partial \tilde{x}_j} = \frac{1}{\lambda_j} \frac{\partial}{\partial x_j}.$$

Now we want to introduce the new complex coordinate variable into the governing equations with the above equation in mind. The summation convection is abandoned in the remaining of this section. Thus, the equation of motion becomes

$$-\rho \omega^2 u_i = \sum \frac{1}{\lambda_j} \frac{\partial T_{ij}}{\partial x_j},$$

and the constitutive equation becomes

$$T_{ij} = c_{ijkl} S_{kl},$$

$$S_{ij} = \frac{1}{2} \left[ \frac{1}{\lambda_j} u_{i,j} + \frac{1}{\lambda_j} u_{j,i} \right].$$

These new forms of the equation of motion and the constitutive equation admit a body-wave solution of the form

$$u(x) = qe^{i(k \cdot \tilde{x})},$$
where \( q = \pm x \), and \( x \) is the unit vector in the propagating direction. The key to remember is that \( \lambda \) will have a complex value only in the absorbing PML region. Therefore, at the interface of the computational domain and the PML, the wave equation will have the same form. In addition, the value of \( \lambda \) will be the same at the interface; thus, any propagating wave will pass through the interface without generating any reflected wave. Hence, the name of perfectly matched layers.

The only item left is the choice of the \( \lambda_j \). Recall the \( \lambda_j = 1 - i\gamma \) is complex and its imaginary part is related to the wave attenuation coefficient; thus, waves in the absorbing PML region are attenuated [7]. To avoid the artificial reflection from the PML interface, we choose the imaginary part of \( \lambda_j = 1 - i\gamma \) to increase gradually; hence,

\[
\gamma = \frac{\omega_j}{\omega},
\]

where

\[
\omega_j = \begin{cases} 
2\pi a_0 f_0(l_{xi}/L_{PML}); & \text{in PML} \\
0; & \text{in computational domain} 
\end{cases}
\]

Here \( a_0 \) is arbitrary and chosen large enough to ensure total wave absorption in the PML, \( f_0 \) is the dominant frequency, \( l_{xi} \) is the length from the interface to the PML member, and \( L_{PML} \) is the total length of the PML region. The PML region is constructed of several PML members.

The new governing equations for the PML region can easily be implemented into the Finite Element Method (FEM). Normally, one rewrites the governing equations in the form:

\[-\rho \omega^2 \lambda_x \lambda_y \lambda_z u = \nabla \cdot \tilde{T},\]

and

\[ \tilde{T} = \tilde{C} : \nabla u, \]

where

\[ \tilde{T} = \nabla : \begin{bmatrix} \lambda_y \lambda_z & 0 & 0 \\ 0 & \lambda_z \lambda_x & 0 \\ 0 & 0 & \lambda_x \lambda_y \end{bmatrix} T, \]

and

\[ \tilde{c}_{ijkl} = c_{ijkl} \frac{\lambda_x \lambda_y \lambda_z}{\lambda_i \lambda_k}, \quad (i, j, k, l = x, y, z). \]

Notice the governing equations and the wave equation in the PML have the same form as the equations in the computational domain. Therefore, the numerical finite element equation are derived with the use of the same stress or velocity variables by the principle of minimum potential energy.

In concluding, the PML method can achieve high accuracy even with small bounded absorbing domains. This reduces the computation cost. The implementation in terms of the FEM is fairly straightforward, since the governing equations for the PML and the computational domain are the same. However, the method is more complicated if the incident wave is outside the PML. Another negative aspect is that for some anisotropic media, intrinsic instabilities appear [7].

5 Infinite Elements

The final method discussed in this report is the concept of infinite elements. The basic idea is to place elements with a special shape function for the geometry at the infinite boundary. Therefore,
there will be two sets of shape functions, the standard shape functions (N) and a growth shape functions (M). The growth shape functions (M) grow without limit as a coordinate approaches infinity, and are applied to the geometry. The standard shape functions (N) are applied to the field variables.

A classic example is of the line element shown in figure (8). In this example the geometry of the element is interpolated as

\[ x = M_1 x_1 + M_2 x_2 \]

\[ M_1 = -\frac{2\xi}{1-\xi}, \quad M_2 = \frac{1+\xi}{1-\xi}. \]

This yields \( x = x_1 \) at \( \xi = -1 \), \( x = x_2 \) at \( \xi = 0 \), and

\[ x_3 = \lim_{\xi \to 1^-} \frac{-2\xi x_1 + (1 + \xi) x_2}{1 - \xi} = \infty. \]

The formation of the property matrices (i.e. the stiffness matrix) proceeds in the standard fashion, except the mapping functions \( M_1 \) and \( M_2 \) are used to form the Jacobian (\( J \)).

In summary the computer implementation of infinite elements is simple. The growth shape functions (M) are used to generate the Jacobian matrix, its inverse, and its determinant. Infinite elements work well in the frequency domain for dynamic analysis. The major disadvantage is that infinite elements have only worked in the time domain when the media and the boundary completely surround the structure, as in a submarine analysis.

6 Paper Summary

Four typical methods for applying a silent boundary for an infinite media have been presented. Each method has its advantageous and disadvantageous. The purpose of the paper was to provide a general background for the different methods. Finally, it should be mentioned that there are other methods.
References


