10

Structure-Structure Interaction
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This Chapter derives characteristic polynomials for time-discrete second-order equations expressed in operational form. These polynomials form the basis for stability analysis of both monolithic and partitioned analysis procedures for coupled systems.

§10.1. The Need for Model Systems

The characteristic matrix equations derived in Chapter 9 may look frightening to an unprepared reader. Consider, for example, a second order system with one million equations, and apply a 3-step integration method. Then $P(z)$ has $2 \times 1,000,000 \times 3 = 6$ million or more eigenvalues (depending on the computational path and the predictor chosen). To verify numerical stability, the location of each eigenvalue would have to be determined. This is further complicated by the fact that the constituent matrices are generally unsymmetric and that in method design it is necessary to work with free parameters.

The presence of this root morass would pose no problem if a simultaneous solution scheme, whether implicit or explicit, is used. The normal modes of the semidiscrete (time-continuous) system survive in the time-discrete (difference) system. One can therefore look at a one-degree-of-freedom model problem, and infer from its spectral analysis the numerical stability of the entire difference system.

Normal modes do not generally survive, however, in the difference system if a partitioned integration scheme is used. This has two implications: (i) the full coupled system has to be considered in the stability evaluation, and (ii) more sophisticated mathematical techniques, such as the theory of positive-real polynomials, must be resorted to in order to arrive at general conclusions.

In the following stability analysis examples a compromise is struck. A model system is still used, but it is no longer one-dimensional: it contains as many equations as partitions. This makes the stability analysis feasible on a CAS while keeping free parameters.

§10.2. Structure Partitions

Consider a structural system governed by the semidiscrete equations of motion

$$M \ddot{u} + C \dot{u} + Ku = f(t).$$  \hspace{1cm} (10.1)

For the ensuing development of a staggered solution procedure, this system is assumed to be partitioned into two fields: $X$ and $Y$, with state vectors $u_x$ and $u_y$, respectively, that interact as sketched in Figure 10.1. The partitioned equations of motion are

$$\begin{bmatrix} M_{xx} & 0 \\ 0 & M_{yy} \end{bmatrix} \begin{bmatrix} \ddot{u}_x \\ \ddot{u}_y \end{bmatrix} + \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix} \begin{bmatrix} \dot{u}_x \\ \dot{u}_y \end{bmatrix} + \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}. \hspace{1cm} (10.2)$$

Note that a block diagonal mass matrix is assumed. In other words, coupling mass terms $M_{xy}$ and $M_{yx}$ must vanish. This can always be accomplished by linking the coupled structures through forces, or via massless connection elements. In addition, both $M_{xx}$ and $M_{yy}$ are assumed to be symmetric and positive definite. The stiffness matrix $K$ is assumed to be symmetric and nonnegative.
§10.2.1. Staggered Partition

A staggered solution scheme with prediction on \( u_x \) is obtained through the following rearrangement of (10.2):

\[
\begin{bmatrix}
M_{xx} & 0 \\
0 & M_{yy}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_x \\
\ddot{u}_y
\end{bmatrix}
+ \begin{bmatrix}
C_{xx} & 0 \\
C_{yx} & C_{yy}
\end{bmatrix}
\begin{bmatrix}
\dot{u}_x \\
\dot{u}_y
\end{bmatrix}
+ \begin{bmatrix}
K_{xx} & 0 \\
K_{yx} & K_{yy}
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix}
= \begin{bmatrix} f_x - C_{xx} \ddot{u}_x^p - K_{xx} u_x^p \\
f_y - K_{yy} u_y^p
\end{bmatrix}.
\]

(10.3)

Because structural systems are usually lightly damped, further investigation of the stability of the staggered procedure will focus on the undamped system

\[
\begin{bmatrix}
M_{xx} & 0 \\
0 & M_{yy}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_x \\
\ddot{u}_y
\end{bmatrix}
+ \begin{bmatrix}
K_{xx} & 0 \\
K_{yx} & K_{yy}
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix}
= \begin{bmatrix} f_x - K_{xx} u_x^p \\
f_y
\end{bmatrix}.
\]

(10.4)

§10.2.2. The Test System

To construct a test system for stability analysis of (10.4), the applied forcing terms in are dropped, and the following vibration eigenproblems are considered:

\[
M_{xi} z_{xi} = \omega_{xi}^2 K_{xi} z_{xi}, \quad M_{yy} z_{yj} = \omega_{yj}^2 K_{yy} z_{yj},
\]

(10.5)

where indices \( i \) and \( j \) run over the number of freedoms in partitions X and Y, respectively.

The eigenproblems (10.5) are called the uncoupled eigensystems since they arise if the coupling terms \( K_{xy} \) and \( K_{yx} \) vanish. The uncoupled squared frequencies \( \omega_{xi}^2 \) and \( \omega_{yj}^2 \) are zero or positive. If \( \omega_{xi}^2 = 0 \), \( z_{xi} \) is a rigid-body mode for the uncoupled partition X, and likewise for Y. The eigenvectors are normalized to unit generalized mass: \( v_x^T M_{xx} v_x = 1 \) and \( v_y^T M_{yy} v_y = 1 \).

We pass to modal coordinates through

\[
\begin{align*}
u_x &= V_x x, \\
u_y &= V_y y.
\end{align*}
\]

(10.6)

where \( x \) and \( y \) collect modal amplitudes. These are substituted in (10.4) and the equations premultiplied by \( V_x^T \) and \( V_y^T \) to produce a modal system of the form

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\ddot{y}
\end{bmatrix}
+ \begin{bmatrix}
\Omega_{xx} & 0 \\
\Omega_{yx} & \Omega_{yy}
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix}
= \begin{bmatrix} -\Omega_{xy} y^p \\
0
\end{bmatrix}.
\]

(10.7)

Here \( \Omega_{xx} \) and \( \Omega_{yy} \) are diagonal matrices formed by the \( \omega_{xi}^2 \) and \( \omega_{yj} \) squared frequencies, respectively. On the other hand \( \Omega_{xy} \) and \( \Omega_{yx} = \Omega_{xy}^T \) are generally full matrices, since the modes of the structure partitions are not necessarily modes of the coupled system. For convenience the \( \{i, j\}^{th} \) entry of \( \Omega_{xy} \) will be written \( \Omega_{xy ij} = \kappa_{ij} \omega_i \omega_j \), in which \( |\kappa_{ij}| \leq 1 \) is the coupling coefficient between partition modes \( i \) and \( j \).\(^1\) Now if (10.8) is stable for any mode pair \( \{x_i, y_j\} \) so is the response of (10.4) since the response in physical coordinates \( x \) and \( y \) is a discrete linear combination as per (10.6). We therefore can restrict consideration to a typical mode pair. Supressing the indices \( i \) and \( j \) for brevity we arrive at the test system

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\ddot{y}
\end{bmatrix}
+ \begin{bmatrix}
\omega_x^2 & 0 \\
\kappa \omega_x \omega_y & \omega_y^2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix} -\kappa \omega_x \omega_y y^p \\
0
\end{bmatrix}.
\]

(10.8)

in which \( \omega_x \geq 0, \omega_y \geq 0 \) and \( |\kappa| \leq 1 \). The last condition comes from the assumption that \( K \) is nonnegative.

\(^1\) If \( \omega_i = 0 \) or \( \omega_j = 0 \) then \( \Omega_{xy ij} = 0 \); else \( K \) would be indefinite. In that case one can take \( \kappa_{ij} = 0 \) for convenience.
10.3 Stability Analysis

The stability analysis of the test system (10.8) confirms the general results of Chapter 9. For specificity the Trapezoidal Rule defined by the operators \( \rho = 1 - 1/z, \sigma = 1 + z \) and \( \delta = \frac{1}{2} h \) was used on both \( u \) and \( v \). For the test system treated with TR the optimal stability polynomial is

\[
\det(P^*(z)) = \det(\rho^2 M + \delta^2 \sigma^2 K + (e^* - 1) \delta^2 \xi K_2) = \frac{1}{16z^4}[16(-1 + z)^4 + h^4(1 + z)^2 ((1 + z)^2 - 4z\kappa^2)\omega_1^2\omega_2^2 + h^2(4(z^2 - 1)^2\omega_1^2 + 4(z - 1)^2(1 + z)^2\omega_2^2)]
\]

(10.9)

Here \( e^* \) is the optimal predictor as listed in Table 9.2, and \( \xi \) is \( \rho, \rho \sigma, \sigma \) and \( \sigma^2 \) for paths \((0'), \(0), \(1) \) and \(2) \), respectively. To convert to an amplification polynomial, simply rationalize via multiplication by \( z^4 \). The Hurwitz polynomial, obtained on replacing \( s = (z - 1)/(z + 1) \) and taking its numerator, is

\[
P(s) = h^4 \omega_1^2\omega_2^2 - h^4\kappa^2\omega_1^2\omega_2^2 + (4h^2\omega_1^2 + 4h^2\omega_2^2 + h^4\kappa^2\omega_1^2\omega_2^2)s^2 + 16s^4.
\]

(10.10)

The only nontrivial stability condition is

\[
h^4 \omega_1^2\omega_2^2 - h^4\kappa^2\omega_1^2\omega_2^2 \geq 0.
\]

(10.11)

This is satisfied for any \( h \) if \( |\kappa| \leq 1 \), which is guaranteed from the construction of the test system and the definiteness conditions on the original governing equations.

The symbolic verification that indeed (10.9) is obtained according to theory is done with the script listed in Figure 10.2, which loops over four computational paths and adjusts the predictor operator.
accordingly. The script calls auxiliary modules described in Chapter 4. The characteristic polynomial is compared against (10.9) and all paths verify coalescence.

In summary, the treatment of structure-structure interaction of undamped or Rayleigh damped systems by staggered methods can be considered as being in good shape. Unconditional stability is attained by carefully choosing the predictor in accordance to the computational path. (As remarked before, equivalent stability does not mean equivalent accuracy; that property has to be analyzed later by the method of modified equations outlined in Chapter 5.)

For generally damped structural systems no general theory is available, and the search for stable partitioned procedures still has to proceed on a case by case basis. However the results of Chapter 9 can be used as a starting point. This suggestion is in fact used in the next Chapter, which deals with control-structure interaction.

Notes and Bibliography

Most of the material in this Chapter has been extracted from portions of an unpublished report [10.18], as well as from [10.20]. At that time stability tests were verified numerically since no computer algebra systems were available.

References


Grcar, J., Birthday of Modern Numerical Analysis, sepp@california.sandia.gov, 2003


§10. References


