Three-Dimensional Linear Elastostatics
The 3D Elasticity Problem

\[ n: \text{exterior normal to } S \]

\[ S_t: \sigma_n = \hat{t} \]

\[ S_u: u = \hat{u} \]

Volume \( V \)
Data Fields in Detail

\[ S_n : \sigma_n = \hat{t} \]

\[ S_u : u = \hat{u} \]

body forces \( \mathbf{b} \) in volume \( V \)
# Governing Equations of Elastostatics

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Valid</th>
<th>Compact Tensor Form</th>
<th>Matrix Form</th>
<th>Component (Indicial) Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>KE</td>
<td>in $V$</td>
<td>$\epsilon = \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \mathbf{D} \cdot \mathbf{u}$</td>
<td>$\mathbf{e} = \mathbf{D} \mathbf{u}$</td>
<td>$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$</td>
</tr>
<tr>
<td>CE</td>
<td>in $V$</td>
<td>$\sigma = \mathbf{E} \cdot \mathbf{e}$</td>
<td>$\sigma = \mathbf{E} \mathbf{e}$</td>
<td>$\sigma_{ij} = E_{ijk\ell} e_{k\ell}$</td>
</tr>
<tr>
<td>BE</td>
<td>in $V$</td>
<td>$\nabla \cdot \sigma + \mathbf{b} = \mathbf{0}$</td>
<td>$\mathbf{D}^T \sigma + \mathbf{b} = \mathbf{0}$</td>
<td>$\sigma_{ij} + b_i = 0$</td>
</tr>
<tr>
<td>PBC</td>
<td>on $S_u$</td>
<td>$\mathbf{u} = \mathbf{\hat{u}}$</td>
<td>$\mathbf{u} = \mathbf{\hat{u}}$</td>
<td>$u_i = \hat{u}_i$</td>
</tr>
<tr>
<td>FBC</td>
<td>on $S_t$</td>
<td>$\sigma \cdot \mathbf{n} = \sigma_n = \mathbf{t} = \mathbf{\hat{t}}$</td>
<td>$\mathbf{P}_n \sigma = \sigma_n = \mathbf{t} = \mathbf{\hat{t}}$</td>
<td>$\sigma_{ij}n_j = \sigma_{ni} = t_i = \hat{t}_i$</td>
</tr>
</tbody>
</table>
The Tonti Diagram - Generic Form

This "dual" part of the Tonti diagram is not used in this course.
The Strong Form of the Tonti Diagram for Elastostatics
(Equations written in indicial form)

\[ \sigma_{ij} = E_{ijkl} e_{kl} \quad \text{in } V \]

\[ \sigma_{ij, j} + b_i = 0 \quad \text{in } V \]

\[ \sigma_{ij} n_j = \hat{t}_i \quad \text{on } S_t \]

\[ u_i = \hat{u}_i \quad \text{on } S_u \]

\[ e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \text{in } V \]
Constructing A Variational Form in Five (not so easy) Steps

<table>
<thead>
<tr>
<th>Step*</th>
<th>Operation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Select master field(s)</td>
<td>One or more of the unknown internal fields (displacements, strains and stresses in the elasticity problem) are chosen as masters.</td>
</tr>
<tr>
<td>2</td>
<td>Select weak connections</td>
<td>Selected strong links are weakened. Slave fields may have to be split if necessary. If so, duplicated slave fields are connected by weak links.</td>
</tr>
<tr>
<td>3</td>
<td>Construct Weak Form (WF)</td>
<td>A Weak Form is established by choosing weights for the weak links and integrating over their domains (volume or surface).</td>
</tr>
<tr>
<td>4</td>
<td>Replace weights by alleged variations and homogenize</td>
<td>WF weight functions are replaced by master field variations propagated through the strong links as appropriate. Master field variations are homogenized using integration by parts (the divergence theorem in 2D or 3D) as necessary.</td>
</tr>
<tr>
<td>5</td>
<td>Find Variational Form (VF)</td>
<td>The VF is obtained as the functional whose first variation reproduces the varied-WF of Step 4. This process may involve trial and error, e.g., playing with alleged variations and their signs. If no VF can be found, one may have to be content with the WF of Step 3 as basis to develop numerical approximations.</td>
</tr>
</tbody>
</table>

* Steps 4-5 are typically the most difficult and time-consuming ones. There is no guarantee that Step 5 will be successful within the Standard Variational Calculus. It becomes always possible through nonstandard extensions such as adjoint variational principles, restricted variational principles, or use of noncommutative variations; none of which are considered here.
Weak Form Departure Point to Derive the Primal (TPE) Functional of Elastostatics

\[ \int_V (\sigma_{ij} e_{ij}, j + b_i) \, dV = 0 \]

\[ \int_S (\sigma_{ij} n_j, i) \, dS = 0 \]

Symbol \( \lambda \) used here for the BE-residual suggests a Lagrange multiplier. That is how some authors construct variational forms by "relaxing" strong links. Scheme first used by Friedrichs (cf. Courant-Hilbert book) and popularized in mechanics & FEM by Fraeijs de Veubeke.

**KE:** \( e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \) in \( V \)

**PBC:** \( u_i = \hat{u}_i \) on \( S_u \)

**BE:** \( \int_V (\sigma_{ij} + b_i) \lambda_i \, dV = 0 \)

**CE:** \( \sigma_{ij} = E_{ijkl} e_{kl} \) in \( V \)

**FBC:** \( \int_{S_i} (\sigma_{ij} n_j - \hat{t}_i) \delta u_i \, dS = 0 \)
Derivation of the Primal (TPE)
Functional of Elastostatics: Step 1

Pick master field: displacements $u_i$

Pick strong and weak links

Strong: $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ in $V$, $\sigma_{ij} = E_{ijkl} e_{k\ell}$ in $V$, $u_i = \hat{u}_i$ on $S_u$.

Weak: $\sigma_{ij,j} + b_i = 0$ in $V$, $\sigma_{ij} n_j = \hat{t}_i$ on $S_t$. 
Derivation of the Primal (TPE) Functional of Elastostatics: Step 2

Write BE as integral statement with 3-vector residual weight

\[ \int_V (\sigma_{ij,j}^u + b_i) \lambda_i \, dV = 0 \]

Apply (tensor form of) Divergence Theorem:

\[ \int_V \sigma_{ij,j}^u \lambda_i \, dV = - \int_V \sigma_{ij}^{u\lambda} \lambda_i \, dV + \int_S \sigma_{ij}^{n\lambda} n_j \lambda_i \, dS \]

\[ \int_V \sigma_{ij,j}^u \lambda_i \, dV = - \int_V \sigma_{ij}^{u\lambda} \frac{1}{2}(\lambda_{i,j} + \lambda_{j,i}) \, dV + \int_S \sigma_{ij}^{n\lambda} n_j \lambda_i \, dS \]

Replace \( \lambda_i \) by displacement variation

\[ \int_V \sigma_{ij,j}^u \delta u_i \, dV = - \int_V \sigma_{ij}^{u\delta e} \delta e_{ij} \, dV + \int_S \sigma_{ij}^{n\delta u} n_j \delta u_i \, dS \]

in which

\[ \delta e_{ij}^u = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \]
Derivation of the Primal (TPE) Functional of Elastostatics: Step 3

Replace previous results into BE integral statement

\[
\int_V \sigma_{ij}^u \delta e_{ij}^u \, dV - \int_V b_i \, \delta u_i \, dV - \int_S \sigma_{ij}^u n_j \, \delta u_i \, dS = 0
\]

Split surface integral and enforce strong connection on \( S_u \)

\[
\int_S \sigma_{ij}^u n_j \, \delta u_i \, dS = \int_{S_t} \sigma_{ij}^u n_j \, \delta u_i \, dS + \int_{S_u} \sigma_{ij}^u n_j \, \delta \hat{u}_i^0 \, dS = \int_{S_t} \sigma_{ij}^u n_j \, \delta u_i \, dS
\]

Treat the FBC weak connection with \( \delta u_i \) as weight

\[
\int_{S_t} (\sigma_{ij}^u n_j - \hat{t}_i) \, \delta u_i \, dS = 0 \quad \text{whence} \quad \int_{S_t} \sigma_{ij}^u n_j \, \delta u_i \, dS = \int_{S_t} \hat{t}_i \, \delta u_i \, dS
\]
Derivation of the Primal (TPE) Functional of Elastostatics: Step 4

State as first variation of an (alleged) functional

\[ \delta \Pi_{\text{TPE}} = \int_V \sigma_{ij} \delta e_{ij}^{\mu} \, dV - \int_V b_i \delta u_i \, dV - \int_S \hat{t}_i \delta u_i \, dS = 0 \]

This is indeed the first variation of

\[ \Pi_{\text{TPE}}[u_i] = \frac{1}{2} \int_V \sigma_{ij}^{\mu} e_{ij}^{\mu} \, dV - \int_V b_i u_i \, dV - \int_S \hat{t}_i u_i \, dS \]

which is the **Total Potential Energy functional** of elastostatics
Split Form of the TPE Functional

\[ \Pi_{\text{TPE}} = U_{\text{TPE}} - W_{\text{TPE}} \]

in which

\[ U_{\text{TPE}} = \frac{1}{2} \int_V \sigma_{ij}^\varepsilon e_{ij}^\varepsilon dV \]
\[ W_{\text{TPE}} = \int_V b_i u_i dV + \int_{S_t} \hat{t}_i u_i dS \]

Physical interpretation:

\( U \) is the strain energy stored in the body, which for elasticity is the internal energy

\( W \) is the work of the applied external forces; which is the same as the negated loads potential: \( W = -V \)