Decomposition of Poisson Problems
Tonti Decomposition

What it is:

Graphical representation of Strong Form (SF) using two auxiliary variables
- intermediate variable
- flux variable
in addition to the primary variable

Advantages:

Visualization of WF and VF derivations
Clarification of BCs
The Poisson Equation

General form for an isotropic medium
\[ \nabla \cdot (k \nabla u) = s \]

If \( k \) is constant (homogeneous medium)
\[ k \nabla^2 u = s \]

If source \( s = 0 \) vanishes, it reduces to Laplace's equation
\[ \nabla^2 u = 0 \]
The Poisson Equation (Cont'd)

Written in full (in 1D, 2D, 3D):

\[
\frac{\partial}{\partial x_1} \left( k \frac{\partial u}{\partial x_1} \right) = s \\
\frac{\partial}{\partial x_1} \left( k \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( k \frac{\partial u}{\partial x_2} \right) = s \\
\frac{\partial}{\partial x_1} \left( k \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( k \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left( k \frac{\partial u}{\partial x_3} \right) = s
\]

For uniform \( k \):

\[
k \frac{\partial^2 u}{\partial x_1^2} = s, \quad k \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) = s, \quad k \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) = s
\]
The Poisson's Equation Models Many Important Steady-State Problems in Engineering & Physics

- **Linear (Fourier Law) Heat Conduction** (Notes, S2.3)
- **Potential Flow** (AVVM Notes, S6.4)
- **Electrostatics** (AVVM Notes, S6.5)
- **Magnetostatics** (AVVM, S6.6, vector form of eq)
- **Bar & Cables** (Exercise 6.2, 1D form)
- **St. Venant's Torsion** (Exercise 6.4, 2D form)
- **Laterally loaded membranes** (2D form)
- **Flow in Porous Media** (Oil and gas industry)
- **Geomechanics** (Earth gravity, mineral prospecting)
The Heat Conduction Problem

Heat source production in $V$: $h$ specified per unit of volume

$S_T: T = \hat{T}$

$S_q: \hat{q}_n = q$

Volume $V$
Heat Conduction: Field Equations
(in full component notation)

**Kinematic equation (KE):** defines temperature gradient

\[
\begin{bmatrix}
g_1 \\
g_2 \\
g_3 \\
\end{bmatrix}
= -k
\begin{bmatrix}
\frac{\partial T}{\partial x_1} \\
\frac{\partial T}{\partial x_2} \\
\frac{\partial T}{\partial x_3} \\
\end{bmatrix}
\]

**Constitutive equation (CE):** Fourier's law of heat conduction (isotropic body)

\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
\end{bmatrix}
= -k
\begin{bmatrix}
g_1 \\
g_2 \\
g_3 \\
\end{bmatrix}
\]

**Balance equation (BE):** heat flux equilibrium

\[
\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} + h = 0
\]
Heat Conduction: Boundary Conditions

**Primary Boundary Condition (PBC):** specified boundary temperature

\[ T = \hat{T} \quad \text{on } S_T \]

**Flux Boundary Condition (FBC):** specified boundary heat flux

\[ q_1 n_1 + q_2 n_2 + q_3 n_3 = \hat{q}_n \quad \text{on } S_q \]

These are the classical BCs for Heat Conduction. Radiation and Convection BCs (which are generally nonlinear) are not considered here.
Heat Conduction: Summary of Governing Equations

<table>
<thead>
<tr>
<th>grad/div notation</th>
<th>indicial (component) notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>KE: ( \nabla T = g ) in ( V ),</td>
<td>KE: ( T_{,i} = g_i ) in ( V ),</td>
</tr>
<tr>
<td>CE: ( -kg = q ) in ( V ),</td>
<td>CE: ( -kg_i = q_i ) in ( V ),</td>
</tr>
<tr>
<td>BE: ( \nabla \cdot q + h = 0 ) in ( V ),</td>
<td>BE: ( q_{i,i} + h = 0 ) in ( V ),</td>
</tr>
<tr>
<td>PBC: ( T = \hat{T} ) on ( S_T ),</td>
<td>PBC: ( T = \hat{T} ) on ( S_T ),</td>
</tr>
<tr>
<td>FBC: ( q \cdot n = q_n = \hat{q}_n ), on ( S_q ).</td>
<td>FBC: ( q_i n_i = q_n = \hat{q}_n ), on ( S_q ).</td>
</tr>
</tbody>
</table>
The Tonti Diagram for the Field Equations of Heat Conduction

\[ g = \text{grad} \ T \ \text{in} \ V \]

\[ q = - k \ g \ \text{in} \ V \]

\[ \text{div} \ q + h = 0 \ \text{in} \ V \]
The Extended Tonti Diagram for the Governing Equations of Heat Conduction

\[ g = \text{grad} \, T \text{ in } V \]

\[ q = -k \, g \text{ in } V \]

\[ \text{div } q + h = 0 \text{ in } V \]

\[ q \, n = q \text{ on } S_q \]

\[ \hat{q} \]

\[ \hat{T} \]

Unknown field

Data field

Strong connection
The Components of the Extended Tonti Diagram for a Strong Form

- Specified primary variable
- Primary variable
- Source function
- Intermediate variable
- Flux variable
- Specified flux variable

Primary boundary conditions
Kinematic equations
Constitutive equations
Balance or equilibrium equations
FIELD EQUATIONS
Flux boundary conditions