A TEMPLATE TUTORIAL

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Dedicated to Pål Bergan on his 60th birthday

Abstract. This article has a dual theme: historical and educational. It is a tutorial on finite element templates for two-dimensional structural problems. The exposition is aimed at readers with introductory level knowledge of finite element methods. It focuses on the four-node plane stress element of flat rectangular geometry, called the “rectangular panel” for brevity. This is one of the two oldest continuum structural elements. On the other hand the concept of finite element templates is a recent development, which had as key source the 1984–87 collaboration between Pål Bergan and the writer on the Free Formulation. Interweaving the old and the new throws historical perspective into the evolution of finite element methods. Templates provide a framework in which diverse element formulations can be fitted, compared and traced back to the sources. On the technical side templates facilitate the unified implementation of element families, as well as the construction of custom elements. To illustrate customization power beyond the rectangle, the Appendix presents the construction of a four-noded bending-optimal trapezoid. This model sidesteps MacNeal’s limitation theorem in that it passes the patch test for any geometry while staying bending-optimal along one direction and retaining full rank.

Key words: finite elements, history, templates, families, clones, quadrilateral membrane, Free Formulation, patch test

1 INTRODUCTION

This article interweaves historical and educational themes with the presentation of a new methodology for finite element development. It is expository in nature, and focuses on “FEM by its history.” It gently exposes the reader to the concept of templates, and explains why they emerge naturally from historical evolution. Much of the material is tutorial in nature, with some extracted from a FEM course offered by the writer.

Templates are parametrized algebraic forms that provide a continuum of consistent and stable finite element models of a given type and node/freedom configuration. Template instances produced by setting values to free parameters furnish specific elements. If the template embodies all possible consistent and stable elements of a given type and configuration, it is called universal.
Befitting the tutorial aim, the main body of this article focuses on the simplest two-dimensional element that possesses a nontrivial template: the four-node plane stress element of flat rectangular geometry. [The three-node linear triangle is simpler but its template is trivial.] This is called the rectangular panel for brevity.

The rectangular panel is interesting from both historical and instructional viewpoints because:
1. It is one of the two oldest continuum finite elements, the other being the linear triangle.¹
2. Along with its plane strain and axisymmetric cousins, it is the configuration treated by most new methods since the birth of finite elements. As such it represents an in-vivo specimen of FEM evolution over the past 50 years.
3. It is amenable to complete analytical development, even for anisotropic material behavior. This makes the element particularly suitable for homework and project assignments.
4. Analytical forms make the concept of signatures and clones highly visible to students.

The paper is organized as follows. Section 2 is a brief outline of element formulation approaches used from 1950 to date. Section 3 introduces the focus problem. Sections 4–6 follow up on the historical theme by presenting stress, strain and displacement-based models for the rectangular panel.

The concept of template is introduced in Section 7 by calling attention to a common structure that lurks behind the stiffness expressions of stress, strain and displacement models. Template terminology follows as consequence: families, signatures, instances and clones. The role of higher order patch tests in optimality is illustrated in Sections 8 and 9. SRI schemes are presented in Section 10 from the template standpoint to show that this approach naturally leads to correct splittings of the elasticity law. The concept of element families is illustrated in Section 11 using stress hybrid and displacement bubbles as examples. These clearly illuminate the futility of the “enrichment” approaches popular in the late sixties. Section 12 provides numerical examples and Section 13 offers conclusions.

The Appendix presents the construction of a four-noded bending-optimal trapezoid. Although this illustrates the customization power of templates beyond rectangles, the more advanced mathematical tools in use behind the scenes can make the results look like black magic to beginners. The optimal trapezoid partly circumvents MacNeal’s limitation theorem² in that it passes the patch test for arbitrary geometry and material and is bending optimal along one direction, while retaining a bounded condition number and thus fulfilling the inf-sup condition.

A sequel to this article³ covers the generalization to an a optimal quadrilateral of arbitrary shape. The algebraic manipulations are beyond the power of any human to work out by hand over a lifetime, so in retrospect it is not surprising that this model has not been discovered sooner. Fortunately the symbolic work falls, although barely, within the grasp of computer algebra systems on a PC. The final results are surprisingly simple and elegant, even for arbitrary anisotropic material. The construction finishes a quest that has preoccupied FEM investigators over several decades.

2 HISTORICAL SKETCH

This section summarizes the history of structural finite elements since 1950 to date. It functions as a hub for dispersed historical references. Readers uninterested in these aspects should proceed directly to Section 3. For exposition convenience, structural “finiteelementology” may be divided into four generations that span 10 to 15 years each. There are no sharp intergenerational breaks, but noticeable changes of emphasis. The ensuing outline
does not cover the conjoint evolution of Matrix Structural Analysis into the Direct Stiffness Method from 1934 through 1970. This was the subject of a separate essay.4

2.1 G1: The Pioneers

The 1956 paper by Turner, Clough, Martin and Topp,1 henceforth abbreviated to TCMT, is recognized as the start of the current FEM, as used in the overwhelming majority of commercial codes. Along with Argyris’ serial5 they prototype the first generation, which spans 1950 through 1962. A panoramic picture of this period is available in two textbooks.6,7 Przemieniecki’s text is still reprinted by Dover. The survey by Gallagher8 was influential but is now difficult to access.

The pioneers were structural engineers, schooled in classical mechanics. They followed a century of tradition in regarding structural elements as a device to transmit forces. This “element as force transducer” was the standard view in pre-computer structural analysis. It explains the use of flux assumptions to derive stiffness equations. Element developers worked in, or interacted closely with, the aircraft industry. [One reason is that only large aerospace companies were then able to afford mainframe computers.] Accordingly they focused on thin structures built up with bars, ribs, spars, stiffeners and panels. Although the Classical Force method dominated stress analysis during the fifties,9 stiffness methods were kept alive by use in dynamics and vibration.

2.2 G2: The Golden Age

The next period spans the golden age of FEM: 1962–1972. This is the “variational generation.” Melosh9 showed that conforming displacement models are a form of Rayleigh-Ritz based on the minimum potential energy principle. This influential paper marks the confluence of three lines of research: Argyris’ dual formulation of energy methods,5 the Direct Stiffness Method (DSM) of Turner,10–12 and early ideas of interelement compatibility as basis for error bounding and convergence.13,14 G1 workers thought of finite elements as idealizations of structural components. From 1962 onward a two-step interpretation emerges: discrete elements approximate continuum models, which in turn approximate real structures.

By the early 1960s FEM begins to expand into Civil Engineering through Clough’s Boeing-Berkeley connection15 and had been named.16,17 Reading Fraeijs de Veubeke’s celebrated article18 side by side with TCMT ¹ one can sense the ongoing change in perspective opened up by the variational framework. The first book devoted to FEM appears in 1967.19 Applications to nonstructural problems start by 1965.20

From 1962 onwards the displacement formulation dominates. This was given a big boost by the invention of the isoparametric formulation and related tools (numerical integration, fitted coordinates, shape functions, patch test) by Irons and coworkers.21–25 Low order displacement models often exhibit disappointing performance. Thus there was a frenzy to develop higher order elements. Other variational formulations, notably hybrids,26–29 mixed30,31 and equilibrium models18 emerged. G2 can be viewed as closed by the monograph of Strang and Fix,32 the first book to focus on the mathematical foundations.

2.3 G3: Consolidation

The post-Vietnam economic doldrums are mirrored during this post-1972 period. Gone is the youthful exuberance of the golden age. This is consolidation time. Substantial effort is put into improving the stock of G2 displacement elements by tools initially labeled “variational crimes” by Strang,33 but later justified. Textbooks by Hughes34 and Bathe35 reflect the technology of this period. Hybrid and mixed formulations record steady progress.36 Assumed strain formulations appear.37 A booming activity in error estimation and mesh adaptivity is fostered by better understanding of the mathematical foundations.38
Commercial FEM codes gradually gain importance. They provide a reality check on what works in the real world and what doesn’t. By the mid-1980s there was gathering evidence that complex and high order elements were commercial flops. Exotic gadgetry interwoven amidst millions of lines of code easily breaks down in new releases. Complexity is particularly dangerous in nonlinear and dynamic analyses conducted by novice users. A trend back toward simplicity starts.39,40

2.4 G4: Back to Basics

The fourth generation begins by the early 1980s. More approaches come on the scene, notably the Free Formulation of Bergan,41,42 which is further discussed below, orthogonal hourglass control,43 Assumed Natural Strain methods,44–47 stress hybrid models in natural coordinates,48–50 as well as variants and derivatives of those approaches: ANDES,51,52 EAS53,54 and others. Although technically diverse the G4 approaches share two common objectives:

(i) Elements must fit into DSM-based programs since that includes the vast majority of production codes, commercial or otherwise.

(ii) Elements are kept simple but should provide answers of engineering accuracy with relatively coarse meshes. These were collectively labeled “high performance elements” in 1989.55

“Things are always at their best in the beginning,” said Pascal. Indeed. By now FEM looks like an aggregate of largely disconnected methods and recipes. Sections 4-6 look at three disparate components of this edifice to anticipate the subsequent exhibition of common features by templates.

2.5 From the Free Formulation to Templates

In the early 1970s Pål Bergan — then a Professor at NTH-Trondheim — and L. Hanssen published a paper56 in MAFELAP II, where a different approach to finite elements was advocated. The concept is well outlined in the Introduction of that paper:

“An important observation is that each element is, in fact, only represented by the numbers in its stiffness matrix during the analysis of the assembled system. The origin of these stiffness coefficients is unimportant to this part of the solution process ... The present approach is in a sense the opposite of that normally used in that the starting point is a generally formulated convergence condition and from there the stiffness matrix is derived ... The patch test is particularly attractive [as such a condition] for the present investigation in that it is a direct test on the element stiffness matrix and requires no prior knowledge of interpolation functions, variational principles, etc.”

This statement sets out what may be called the direct algebraic approach to finite elements: the element stiffness is to be derived directly from consistency conditions — provided by the Individual Element Test of Bergan and Hanssen56,57 — plus stability and accuracy considerations to determine algebraic redundancies if any.

This ambitious goal proved initially elusive because the direct algebraic construction of the stiffness matrix of most multidimensional elements becomes effectively a problem in constrained optimization. In the symbolic form necessitated by element design, such problem is much harder to tackle than the conventional element construction methods. A constructive step was provided by the Free Formulation, which emerged over the next decade.42 The foregoing difficulties were addressed using divide and conquer. The stiffness is decomposed into a basic part that takes care of consistency and mixability, and a higher order (HO) part that takes care of stability (rank sufficiency) and accuracy. Orthogonality conditions enforced between the two parts avoid pollution of the basic element response by the higher order component.

The writer’s acquaintance with the FF began in 1984, while Pål Bergan was spending a 9-month sabbatical at Stanford. At the time the writer was in the staff of the Applied
Mechanics Laboratory of Lockheed Palo Alto Research Laboratories (LPARL, located in the Stanford Industrial Park), involved in gruelling software development supporting the Trident II system and underwater shock analysis codes. Our collaboration, supported by the LPARL Independent Research program, was the first element-related project undertaken by the writer since Berkeley days. It was a welcome respite from the software grind. It resulted in the first rank-sufficient triangular membrane element with drilling freedoms that passed the IET. Tom Hughes kindly speeded up publication.58

The move to the University of Colorado in 1986 allowed the writer to pursue further these ideas within an academic environment. In the FF the HO stiffness was based on a displacement formulation, whereas the basic stiffness is method independent. Other techniques were tried for the HO stiffness. Particularly successful was a modification of the Assumed Natural Strain (ANS) method of Park and Stanley.46,47 In this variant only the deviatoric part of the assumed strains was carried forward, and the ANDES method emerged.51,52 Gradually the realization dawned that all consistent and rank sufficient elements, no matter how they are derived, must fit an algebraic form with free parameters, baptized as template. This “template as umbrella” concept emerged by 1990. It has since developed by fits and starts. The current status of the subject is summarized in Section 13.3

3 PROBLEM DESCRIPTION

3.1 Governing Equations

Consider the thin homogeneous plate in plane stress shown in Figure 1. The inplane displacements are \{u_x, u_y\}, the associated strains are \{e_{xx}, e_{yy}, e_{xy}\} and the inplane (membrane) stresses are \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}. Prescribed inplane body forces are \{b_x, b_y\}, but they will be set to zero in derivations of equilibrium elements. Prescribed displacements and surface tractions are denoted by \{\hat{u}_x, \hat{u}_y\} and \{\hat{t}_x, \hat{t}_y\} respectively. All fields are considered uniform through the thickness \(h\). The governing plane-stress elasticity equations are

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy}
\end{bmatrix} + \begin{bmatrix} 0 \\
0
\end{bmatrix},
\]

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy}
\end{bmatrix} + \begin{bmatrix} b_x \\
b_y
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}.
\]

The compact matrix version of (1) is

\[ e = Du, \quad \sigma = Ee, \quad D^T \sigma + b = 0, \]
in which \( E \) is the plane stress elasticity matrix. Assuming this to be nonsingular, the inverse of \( \sigma = E e \) is

\[
\begin{bmatrix}
  e_{xx} \\
  e_{yy} \\
  2e_{xy}
\end{bmatrix} = \begin{bmatrix}
  C_{11} & C_{12} & C_{13} \\
  C_{12} & C_{22} & C_{23} \\
  C_{13} & C_{23} & C_{33}
\end{bmatrix} \begin{bmatrix}
  \sigma_{xx} \\
  \sigma_{yy} \\
  \sigma_{xy}
\end{bmatrix}, \quad \text{or} \quad e = C\sigma, \tag{3}
\]

where \( C = E^{-1} \) is the matrix of elastic compliances.

### 3.2 The Rectangular Panel

The focus of this tutorial exposition is the rectangular panel. This is depicted in Figure 2. For an individual element the side-aligned local axes are also denoted as \( \{x, y\} \) for brevity. The inplane dimensions are \( a \) and \( b = a/\gamma \), where \( \gamma = a/b \) is the aspect ratio. The thickness and elastic properties are constant over the element. The element has 4 corner nodes and 8 external (connective) degrees of freedom. The node displacement and force vectors are configured as

\[
u = \begin{bmatrix} u_{1x} & u_{1y} & u_{2x} & u_{2y} & u_{3x} & u_{3y} & u_{4x} & u_{4y} \end{bmatrix}^T, \tag{4}
\]

\[f = \begin{bmatrix} f_{1x} & f_{1y} & f_{2x} & f_{2y} & f_{3x} & f_{3y} & f_{4x} & f_{4y} \end{bmatrix}^T. \tag{5}\]

As noted in the Introduction, most of the FEM formulation methods chronicled in Section 2 have been tried on this configuration as well as its plane strain and axisymmetric cousins. The reason for this popularity is that the rectangular panel is the simplest multi-dimensional element that can be improved. [The three-node linear triangle is simpler but cannot be improved.]

In keeping with the expository theme, the next three sections derive the rectangular panel stiffness from stress, strain and displacement assumptions, respectively. Mirroring history, the derivation of stress and strain models follows the matrix-based direct elasticity approach used by the first generation, as summarized in Gallagher’s review. 8

Ironically, the direct derivation from stress modes a la TCMT furnishes optimal or near-optimal elements with little toil, whereas the variationally derived displacement models will need tweaking to become useful.

### 4 THE STRESS ELEMENT

TCMT\(^1\) is the starting point. In a historical summary Clough\(^{15}\) remarks that the paper belatedly reports work performed at Boeing’s Commercial Airplane Division in 1952–53 (indeed a TCMT footnote states that the material was presented at the 22nd Annual Meeting of IAS, held on January 25–29, 1954.) Besides bars, beams and spars, TCMT presents two plane stress elements for modeling wing cover plates: the three-node triangle and the four-node flat rectangular panel. Quadrilateral panels of arbitrary geometry, not necessarily flat, are constructed as assemblies of four triangles.

Readers perusing that article for the first time have a surprise in store. The stiffness properties of both panel elements are derived from stress assumptions, rather than displacements, as became popular in the second generation. More precisely, simple patterns
of interelement boundary tractions (a.k.a. stress flux modes) that satisfy internal equilibrium are taken as starting point. Twenty years later Fraeijs de Veubeke \(^5\) \(^9\) systematically extended the same idea in a variational setting, to produce what he called diffusive equilibrium elements. These are designed to weakly enforce interelement flux conservation. The comedy continues: mathematicians recently rediscovered flux elements, now renamed as “Discontinuous Galerkin Methods,” blissfully unaware of previous work.

The derivation below largely follows Chapter 3 of Gallagher, \(^8\) who presents a step by step procedure for what he calls the “equivalent force” approach. The main extension provided here is allowing for anisotropic material.

### 4.1 The 5-Parameter Stress Field

Since TCMT the appropriate stress field for the rectangular panel is known to be

\[
\begin{align*}
\sigma_{xx} &= \mu_1 + \mu_4 \frac{y}{b}, \\
\sigma_{yy} &= \mu_2 + \mu_5 \frac{x}{a}, \\
\sigma_{xy} &= \mu_3.
\end{align*}
\]

The five \(\mu_i\) are stress-amplitude parameters with dimension of stress. They are collected in the 5-vector

\[
\mu = [\mu_1 \ \mu_2 \ \mu_3 \ \mu_4 \ \mu_5]^T.
\]

The field (6) satisfies the internal equilibrium equations (1) under zero body forces. Evaluation over element sides produces the traction patterns of Figure 3, transcribed verbatim from TCMT. Why five? On p. 813: “These load states are seen to represent uniform and linearly varying stresses plus constant shear, along the plate edges. Later it will be seen that the number of load states must be \(2n - 3\), where \(n = \text{number of nodes}\).”

To establish connection to node displacements, \(\mu\) is extended as

\[
\mu_+ = [\mu_1 \ \mu_2 \ \mu_3 \ \mu_4 \ \mu_5 \ \mu_6 \ \mu_7 \ \mu_8]^T
\]

This array contains three dimensionless coefficients: \(\mu_6, \mu_7\) and \(\mu_8\), which define amplitudes of the three element rigid body modes (RBMs):

RBM\(_1\): \(u_x = \mu_6 a, \ u_y = 0\), RBM\(_2\): \(u_x = 0, \ u_y = \mu_7 b\), RBM\(_3\): \(u_x = -\mu_8 y, \ u_y = \mu_8 x\).

These modes produce zero stress. The foregoing relations may be recast in matrix form:

\[
\sigma = N\mu = N_+\mu_+, \quad N = \begin{bmatrix}
1 & 0 & 0 & \frac{y}{b} & 0 \\
0 & 1 & 0 & 0 & \frac{x}{a} \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}, \quad N_+ = \begin{bmatrix}
1 & 0 & 0 & \frac{y}{b} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \frac{x}{a} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

The boundary traction patterns of Figure 3 are converted to node forces by statics. This yields

\[
f = A\mu, \quad A^T = \frac{1}{2} h \begin{bmatrix}
-b & 0 & b & 0 & b & 0 & -b & 0 \\
0 & -a & 0 & -a & 0 & a & 0 & a \\
-a & -b & -a & b & a & b & a & -b \\
\frac{1}{6}b & 0 & -\frac{1}{6}b & 0 & \frac{1}{6}b & 0 & -\frac{1}{6}b & 0 \\
0 & \frac{1}{6}a & 0 & -\frac{1}{6}a & 0 & \frac{1}{6}a & 0 & -\frac{1}{6}a
\end{bmatrix}.
\]
Matrix $\mathbf{A}$ is the equilibrium matrix, also known as the leverage matrix in the FEM literature. When restricted to constant stress states (the first three columns of $\mathbf{A}$), it is called a force-lumping matrix and denoted by $\mathbf{L}$ in the Free Formulation of Bergan.\textsuperscript{41,42,58,60-65}

### 4.2 The Generalized Stiffness

Integrating the complementary energy density $\mathcal{U}^* = \frac{1}{2} \mathbf{\sigma}^T \mathbf{C} \mathbf{\sigma}$ over the element volume $V$ and identifying $\mathcal{U}^* = \int_V \mathcal{U}^* dV$ with $\frac{1}{2} \mathbf{\mu}^T \mathbf{F}_\mu \mathbf{\mu}$ yields the $5 \times 5$ flexibility matrix $\mathbf{F}_\mu$ in terms of the stress parameters. Its inverse is the generalized stiffness matrix $\mathbf{S}_\mu = \mathbf{F}_\mu^{-1}$:

\[
\mathbf{F}_\mu = V \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 \\
0 & 0 & \frac{1}{12} C_{11} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{12} C_{22} & 0
\end{bmatrix}, \quad \mathbf{S}_\mu = \frac{1}{V} \begin{bmatrix}
E_{11} & E_{12} & E_{13} & 0 & 0 \\
E_{12} & E_{22} & E_{23} & 0 & 0 \\
E_{13} & E_{23} & E_{33} & 0 & 0 \\
0 & 0 & 12 C_{11}^{-1} & 0 & 0 \\
0 & 0 & 0 & 12 C_{22}^{-1} & 0
\end{bmatrix},
\]

in which $V = abh$ is the volume of the element.

### 4.3 The Physical Stiffness

Integration of the slave strain field $\mathbf{e} = \mathbf{E}^{-1} \mathbf{\sigma} = \mathbf{C} \mathbf{N}_+ \mathbf{\mu}_+$ produces the displacement field

\[
\begin{align*}
\mu_x(x, y) &= \mu_6 a + \frac{1}{8} \omega_6 + (\mu_1 C_{11} + \mu_2 C_{12} + \mu_3 C_{13}) x + \left(\frac{1}{2} (\mu_1 C_{13} + \mu_2 C_{23} + \mu_3 C_{33}) - \mu_8\right) y \\
&\quad + \frac{1}{2} (\mu_5/a) C_{12} x^2 + (\mu_4/b) C_{11} x y + \frac{1}{2} (\mu_4/b) C_{12} y^2, \\
\mu_y(x, y) &= \mu_7 b + \frac{1}{8} \omega_7 + \frac{1}{2} (\mu_1 C_{13} + \mu_2 C_{23} + \mu_3 C_{33}) + \mu_8 x + (\mu_1 C_{12} + \mu_2 C_{22} + \mu_3 C_{23}) y \\
&\quad + \frac{1}{2} (\mu_5/a) C_{23} + (\mu_4/b) C_{11}) x^2 + (\mu_5/a) C_{22} y + \frac{1}{2} (\mu_4/b) C_{23} x y^2,
\end{align*}
\]

with $\omega_6 = -b^2 C_{13} \mu_4 / b + (b^2 C_{22} - a^2 C_{12}) (\mu_5 / a)$ and $\omega_7 = (a^2 C_{11} - b^2 C_{12}) (\mu_4 / b) - a^2 C_{23} \mu_5 / a$. The constant terms in $\mu_x$ and $\mu_y$, which do not affect strains and stresses, have been adjusted to get relatively simple terms in columns 4 through 8 of the matrix $\mathbf{T}_+$ below. Physically, (13) aligns the bending deformation patterns along the {x, y} axes. Evaluating (13) at the nodes we obtain the matrix that connects node displacements to stress parameters: $\mathbf{u} = \mathbf{T}_+ \mathbf{\mu}_+$, where

\[
\mathbf{T}_+ = \frac{1}{3} \begin{bmatrix}
-2 a C_{11} - b C_{13} - 2 a C_{12} - b C_{23} - 2 a C_{13} - b C_{33} & a C_{11} & 0 & 4 a & 0 & 2 b \\
-2 b C_{12} - a C_{13} - 2 b C_{22} - a C_{23} - 2 b C_{23} - a C_{33} & 0 & b C_{22} & 0 & 4 b & 2 a \\
2 a C_{11} - b C_{13} & 2 a C_{12} - b C_{23} & 2 a C_{13} - b C_{33} - a C_{11} & 0 & 4 a & 0 & 2 b \\
-2 b C_{12} + a C_{13} - 2 b C_{22} + a C_{23} - 2 b C_{23} + a C_{33} & 0 & - b C_{22} & 0 & 4 b & 2 a \\
2 a C_{11} + b C_{13} & 2 a C_{12} + b C_{23} & 2 a C_{13} + b C_{33} - a C_{11} & 0 & 4 a & 0 & -2 b \\
2 b C_{12} + a C_{13} & 2 b C_{22} + a C_{23} & 2 b C_{23} + a C_{33} & 0 & b C_{22} & 0 & 4 b & 2 a \\
-2 a C_{11} + b C_{13} - 2 a C_{12} + b C_{23} - 2 a C_{13} + b C_{33} - a C_{11} & 0 & 4 a & 0 & 2 b \\
2 b C_{12} - a C_{13} & 2 b C_{22} - a C_{23} & 2 b C_{23} - a C_{33} & 0 & - b C_{22} & 0 & 4 b & 2 a
\end{bmatrix}
\]

The determinant of $\mathbf{T}_+$ is $a^4 b^4 C_{11} C_{22} \det(\mathbf{C})$, so $\mathbf{T}_+$ is invertible if $a \neq 0$, $b \neq 0$, ...
The physical stiffness $K_\sigma$ relates $f = K_\sigma u$, where the $\sigma$ subscript flags the stress

$$
\begin{align*}
C_{11} \neq 0, C_{22} \neq 0 \text{ and } C \text{ is nonsingular. Inversion yields } \mu_+ = U_+ u, \text{ where}
\end{align*}
$$

$$
U_+ = T_+^{-1} = \frac{1}{ab}
\begin{bmatrix}
U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} \\
U_{21} & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} & U_{27} & U_{28} \\
U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} & U_{37} & U_{38} \\
bC_{11} & 0 & -bC_{11} & 0 & bC_{11} & 0 & bC_{11} & 0 \\
0 & aC_{22} & 0 & -aC_{22} & 0 & aC_{22} & 0 & -aC_{22} \\
\frac{1}{4} b & 0 & \frac{1}{4} b & 0 & \frac{1}{4} b & 0 & \frac{1}{4} b & 0 \\
\frac{1}{4} a & -\frac{1}{4} b & \frac{1}{4} a & \frac{1}{4} b & -\frac{1}{4} a & \frac{1}{4} b & -\frac{1}{4} a & \frac{1}{4} b \\
\end{bmatrix},
\end{align*}
$$

in which $U_{11} = -\frac{1}{2}(bE_{11} + aE_{13})$, $U_{12} = -\frac{1}{2}(aE_{12} + bE_{13})$, $U_{13} = \frac{1}{2}(bE_{11} - aE_{13})$, $U_{14} = -\frac{1}{2}(aE_{12} - bE_{13})$, $U_{15} = \frac{1}{2}(bE_{11} + aE_{13})$, $U_{16} = \frac{1}{2}(aE_{12} + bE_{13})$, $U_{17} = -\frac{1}{2}(bE_{11} - aE_{13})$, $U_{18} = \frac{1}{2}(aE_{12} + bE_{13})$, $U_{21} = \frac{1}{2}(bE_{12} + aE_{23})$, $U_{22} = -\frac{1}{2}(aE_{12} - bE_{13})$, $U_{23} = \frac{1}{2}(bE_{12} - aE_{23})$, $U_{24} = \frac{1}{2}(aE_{13} - bE_{23})$, $U_{25} = \frac{1}{2}(bE_{12} + aE_{23})$, $U_{26} = \frac{1}{2}(aE_{22} + bE_{23})$, $U_{27} = -\frac{1}{2}(bE_{12} - aE_{23})$, $U_{28} = \frac{1}{2}(aE_{12} - bE_{13})$, $U_{31} = \frac{1}{2}(bE_{13} + aE_{33})$, $U_{32} = -\frac{1}{2}(aE_{13} + bE_{33})$, $U_{33} = \frac{1}{2}(bE_{13} - aE_{33})$, $U_{34} = \frac{1}{2}(aE_{23} + bE_{33})$, $U_{35} = \frac{1}{2}(bE_{13} + aE_{33})$, $U_{36} = \frac{1}{2}(aE_{23} + bE_{33})$, $U_{37} = -\frac{1}{2}(bE_{13} - aE_{33})$ and $U_{38} = \frac{1}{2}(aE_{23} - bE_{13})$. The stress-displacement matrix $U$ that relates stress parameters to displacements: $\mu = U + u$, is obtained by extracting the first five rows of $U_+$:

$$
U = \frac{1}{ab}
\begin{bmatrix}
U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} \\
U_{21} & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} & U_{27} & U_{28} \\
U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} & U_{37} & U_{38} \\
bC_{11} & 0 & -bC_{11} & 0 & bC_{11} & 0 & bC_{11} & 0 \\
0 & aC_{22} & 0 & -aC_{22} & 0 & aC_{22} & 0 & -aC_{22} \\
\end{bmatrix} = S_\mu A^T.
\end{align*}
$$

The relation $U = S_\mu A^T$ can be checked directly. For this element it can be proven to hold by energy methods, but that was not obvious in 1952. It must have been a relief when the element stiffness came out symmetric. As Gallagher remarks on p. 22, symmetry is the exception rather than the rule for more general geometric configurations. That complication proved a big boost for the energy and variational methods of the second generation.

The physical stiffness $K_\sigma$ relates $f = K_\sigma u$, where the $\sigma$ subscript flags the stress
element. Combining \( f = A\mu \) and \( \mu = Uu = S_\mu A^T u \) yields

\[
K_\sigma = AU = AS_\mu A^T. \tag{17}
\]

Figure 4 summarizes the foregoing derivation steps. Note that one can bypass the calculation of the generalized stiffness \( S_\mu \) if so desired, as diagramed on the left of that figure. This is convenient for presentation to students without a background on energy methods.

Note that the displacement field (13) contains quadratic terms if \( \mu_4 \) or \( \mu_5 \) are nonzero. Hence the element is nonconforming. This is acknowledged but dismissed as innocuous on p. 814 of TCMT.

5 THE STRAIN ELEMENT

A strain-assumed element can be developed through an entirely analogous procedure. The counterpart of (6) is

\[
e_{xx} = \chi_1 + \chi_4 \frac{y}{b}, \quad e_{yy} = \chi_2 + \chi_5 \frac{x}{a}, \quad 2e_{xy} = \chi_3. \tag{18}
\]

where the \( \chi_i \) are dimensionless strain-amplitude parameters. They are collected in the 5-vector

\[
\chi = [\chi_1 \, \chi_2 \, \chi_3 \, \chi_4 \, \chi_5]^T. \tag{19}
\]

An extended vector is constructed by appending the RBM amplitudes

\[
\chi_+ = [\chi_1 \, \chi_2 \, \chi_3 \, \chi_4 \, \chi_5 \, \chi_6 \, \chi_7 \, \chi_8]^T. \tag{20}
\]

in which \( \chi_6, \chi_7 \) and \( \chi_8 \) are defined a a manner similar to (9). Note that \( e = N\chi = N_+\chi_+ \) where \( N \) and \( N_+ \) are defined in (10). Integrating the strains yields the displacement field

\[
u_x(x, y) = \chi_6 + \chi_8y + (\chi_1 + \chi_4/b)xy - \frac{1}{4}(\chi_5/a)y^2,
\]

\[
u_y(x, y) = \chi_7 + (\chi_3 - \chi_8)x + \chi_2y - \frac{1}{4}(\chi_4/b)x^2 + (\chi_5/a)xy. \tag{21}
\]

Evaluating at the nodes and inverting yields \( \chi = B_\chi u \) where

\[
B_+ = \frac{1}{8ab}
\begin{bmatrix}
-4b & 0 & 4b & 0 & 4b & 0 & -4b & 0 \\
0 & -4a & 0 & -4a & 0 & 4a & 0 & 4a \\
-4a & -4b & -4a & 4b & 4a & 4a & -4b \\
8b & 0 & -8b & 0 & 8b & 0 & -8b & 0 \\
0 & 8a & 0 & -8a & 0 & 8a & 0 & -8a \\
2ab & b^2 & 2ab & -b^2 & 2ab & b^2 & 2ab & -b^2 \\
a^2 & 2ab & -a^2 & 2ab & a^2 & 2ab & -a^2 & 2ab \\
-4a & 0 & -4a & 0 & 4a & 0 & 4a & 0
\end{bmatrix} \tag{22}
\]

from which we extract the first five rows to get the strain-displacement matrix relating \( \chi = B_\chi u \):

\[
B_\chi = \frac{h}{2V}
\begin{bmatrix}
-b & 0 & b & 0 & b & 0 & -b & 0 \\
0 & -a & 0 & -a & 0 & a & 0 & a \\
-a & -b & -a & b & a & b & a & -b \\
2b & 0 & -2b & 0 & 2b & 0 & -2b & 0 \\
0 & 2a & 0 & -2a & 0 & 2a & 0 & -2a
\end{bmatrix} \tag{23}
\]
For use below we note the following relation between the transformation matrices of the stress and strain elements

\[
A^T = V D_A B_{\chi}, \quad B_{\chi} = \frac{1}{V} D_A^{-1} A^T, \quad D_A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{12} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 12
\end{bmatrix} \quad = D_A^T.
\]

From (11) the lumping of the slave stress field \( E_{e} = E N_{\chi} \) to node forces can be worked out to be

\[
f = A E_{+} \chi = V B_{\chi}^T D_A E_{+} \chi, \quad \text{with} \quad E_{+} = \begin{bmatrix}
E_{11} & E_{12} & E_{13} & 0 & 0 \\
E_{12} & E_{22} & E_{23} & 0 & 0 \\
E_{13} & E_{23} & E_{33} & 0 & 0 \\
0 & 0 & 0 & E_{11} & 0 \\
0 & 0 & 0 & 0 & E_{22}
\end{bmatrix}
\]

Combining previous equations, the physical element stiffness is

\[
K_e = V B_{\chi}^T D_A E_{+} B_{\chi} = B_{\chi}^T K_{\chi} B_{\chi}, \quad \text{with} \quad K_{\chi} = V D_A E_{+}.
\]

Here \( K_{\chi} \) denotes the generalized stiffness in terms of \( \chi \). This matrix may be obtained also from standard energy arguments: the strain energy density is \( U = \frac{1}{2} \chi^T E \chi \). Integrating over the element volume: \( U = \int_V U dV \) and identifying with \( \frac{1}{2} \chi^T K_{\chi} \chi \) gives

\[
K_{\chi} = V D_A E_{+} = V \begin{bmatrix}
E_{11} & E_{12} & E_{13} & 0 & 0 \\
E_{12} & E_{22} & E_{23} & 0 & 0 \\
E_{13} & E_{23} & E_{33} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{12} E_{11} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{12} E_{22}
\end{bmatrix}
\]

Figure 5 summarizes the foregoing derivation steps. The direct step from \( \chi \) to \( f \) on the left is more difficult to explain to students than the step from \( u \) to \( \mu \) in Figure 4. The energy based formulation shown on the right of Figure 5 tends to be more palatable.
6 THE CONFORMING DISPLACEMENT ELEMENT

This derivation of the assumed-displacement element starts from a conforming displacement field that enforces linear edge displacements. Using the matrix notation of Felippa and Clough for Irons’ isoparametric formulation specialized to the rectangle, the displacement field is bilinearly interpolated as

\[
\begin{bmatrix}
u_x(x, y) \\
u_y(x, y)
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
-a & 0 & a & 0 & -a & 0 \\
0 & -b & 0 & -b & 0 & b
\end{bmatrix} \begin{bmatrix}
\frac{1}{4}(1 - \xi)(1 - \eta) \\
\frac{1}{4}(1 + \xi)(1 - \eta) \\
\frac{1}{4}(1 + \xi)(1 + \eta) \\
\frac{1}{4}(1 - \xi)(1 + \eta)
\end{bmatrix},
\]

(28)

where \(\xi = 2x/a\) and \(\eta = 2y/b\) are the dimensionless quadrilateral coordinates running from \(-1\) to \(1\). The derivation based on the minimum potential energy principle is standard textbook material and only the final result is presented here:

\[
K_u = B_u^T K_q B_u, \quad \text{with} \quad K_q = \frac{1}{V} \begin{bmatrix}
E_{11} & E_{12} & E_{13} & 0 & 0 \\
E_{12} & E_{22} & E_{23} & 0 & 0 \\
E_{13} & E_{23} & E_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{11} & Q_{12} \\
0 & 0 & 0 & Q_{12} & Q_{22}
\end{bmatrix},
\]

(29)

in which \(B_u = A_0^T\) as given by (11) and

\[
Q_{11} = 12 \frac{b^2 E_{11} + a^2 E_{33}}{ab^3 h}, \quad Q_{12} = 12 \left( \frac{E_{13}}{a^2 h} + \frac{E_{23}}{b^2 h} \right), \quad Q_{22} = 12 \frac{a^2 E_{22} + b^2 E_{33}}{a^3 b h}
\]

(30)

This model has a chequered history. It was first derived as a rectangular panel with edge reinforcements (omitted here) by Argyris in his 1954 Aircraft Engineering series; see pp. 49–52 of the Butterworths reprint.5 Argyris used bilinear displacement interpolation in Cartesian coordinates. After much flailing, a conforming generalization to arbitrary geometry was published in 1964 by Taig and Kerr using quadrilateral-fitted coordinates called \(\{\xi, \eta\}\) but running from \(0\) to \(1\). This paper cites an 1961 English Electric Aircraft internal report as original source but Irons and Ahmad25 comment in their reference [108] that the work goes back to 1957.) Irons, who was aware of Taig’s work while at Rolls Royce, created the seminal isoparametric family as a far-reaching extension upon moving to Swansea.21–24

7 TEMPLATES

7.1 Stiffness Decomposition

The stiffnesses \(K_\sigma, K_e\), and \(K_u\) derived in the foregoing three Sections do not appear to have much in common. Indeed if one looks at just the matrix entries no pattern is readily seen. Closer examination reveals, however, that they are instances of the algebraic form

\[
K = K_b + K_h = V H_c^T E H_c + V H_h^T W R W^T R W H_h,
\]

(31)

where \(V = abh\) is the element volume and

\[
H_c = \frac{1}{2ab} \begin{bmatrix}
-b & 0 & b & 0 & b & 0 & -b & 0 \\
0 & -a & 0 & -a & 0 & a & 0 & a \\
-a & -b & -a & b & a & b & a & -b
\end{bmatrix},
\]

\[
H_h = \frac{1}{2} \begin{bmatrix}
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & -1
\end{bmatrix},
\]

(32)

\[
W = \begin{bmatrix}
1/a & 0 \\
0 & 1/b
\end{bmatrix}, \quad R = \begin{bmatrix}
R_{11} & R_{12} \\
R_{12} & R_{22}
\end{bmatrix}.
\]
Matrices $H_c$ and $H_h$ are identical for the three models. The generalized bending rigidity $R$ is formulation dependent. Matrix $W$ is a “higher-order-mode weighting matrix,” hence the notation. For rectangular panels $W$ is diagonal and formulation independent. For more complex geometries discussed in the Appendix and in the sequel $W$ may be formulation-adjusted to make $R$ simpler.

For the stress, strain and displacement models $R$ becomes $R_\sigma$, $R_e$ and $R_u$, respectively, where

$$R_\sigma = \frac{1}{3} \begin{bmatrix} C_{11}^{-1} & 0 & 0 \\ 0 & C_{22}^{-1} & 0 \\ 0 & 0 & C_{33}^{-1} \end{bmatrix}, \quad R_e = \frac{1}{3} \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix}, \quad R_u = \frac{1}{3} \begin{bmatrix} E_{11} + \frac{a^2 E_{33}}{b^2} & \frac{b E_{13}}{a} + \frac{a E_{23}}{b} \\ \frac{b E_{13}}{a} + \frac{a E_{23}}{b} & E_{22} + \frac{b^2 E_{33}}{a^2} \end{bmatrix}$$

But we are not in fact restricted to these. Other expressions for $R$ would yield other $K$.

These are possible, although not necessarily useful, stiffnesses for the rectangular panel if $R$ is symmetric and positive definite, and if its entries have physical dimensions of elastic moduli. Further if $E_{13} = E_{23} = 0$ we set $R_{12} = 0$. The key discovery is that the element formulation affects only part of the stiffness expression, as highlighted in Figure 6.

### 7.2 Template Terminology

The algebraic form characterized by (31) and (32) is called a finite element stiffness template, or template for short.

Matrices $K_b$ and $K_h$ are called the basic and higher-order stiffness matrix, respectively, in accordance with the fundamental decomposition of the Free Formulation. These two matrices play different and complementary roles.

The basic stiffness $K_b$ takes care of consistency and mixability. In the Free Formulation a restatement of (31) is preferred:

$$K_b = V^{-1} L E L^T,$$

where $L = H_c / V$ is called the force lumping matrix, or simply lumping matrix.

The higher order stiffness $K_h$ is a stabilization term that provides the correct rank and may be adjusted for accuracy. This matrix is orthogonal to rigid body motions and constant strain states. To verify the claim for this particular template introduce the following $8 \times 6$
CARLOS A. FELIPPA / A Template Tutorial

RectPanel4TemplateStiffness[{a_,b_},Emat_,Cmat_,h_,name_,Rlist_] :=
Module[{V,found,Hc,Hh,W,Ke}, V=a*b*h;
{WRW,found}=RectPanel4TemplateWRW[{a,b},Emat,Cmat,name,Rlist];
If [Not[found], Print["Illegal elem name: ",name]; Abort[]];
Hc={{-b,0,b,0,b,0,-b,0}, {0,-a,0,-a,0,a,0,a},
{-a,-b,-a,b,a,b,a,-b}}/(2*a*b);
Hh={{1,0,-1,0,1,0,-1,0},{0,1,0,-1,0,1,0,-1}}/2;
Ke=V*Transpose[Hc].Emat.Hc+V*Transpose[Hh].WRW.Hh;
Return[Simplify[Ke]]];

RectPanel4TemplateWRW[{a_,b_},Emat_,Cmat_,name_,Rlist_] :=
Module[{R11,R12,R22,Rmat,E11,E12,E13,E22,E23,E33,
found=False,C11,C22,C33,C12,C13,C23,Edet,Cdet,W,WRW},
{E11,E12,E13}={Emat[[1]]};
If [Length[Cmat]<=0,
C11=(E22*E33-E23^2)/Edet; C22=(E11*E33-E13^2)/Edet;
C33=(E11*E22-E12^2)/Edet; C12=(E13*E23-E12*E33)/Edet;
{{C11,C12,C13},{C12,C22,C23},{C13,C23,C33}}=Cmat;
{{C11,C12,C13},{C12,C22,C23},{C13,C23,C33}}=Cmat]
If [name=="Stress"||name=="QM6"||name=="Q6",
R11=1/(3*C11); R22=1/(3*C22); R12=0; found=True]
If [name=="Strain", R11=E11/3; R22=E22/3; R12=0; found=True]
If [name=="Disp",  R11=(E11+E33*a^2/b^2)/3;
R22=(E22+E33*b^2/a^2)/3; R12=(E13*b+a*E23+a*b)/3; found=True]
If [name=="Arbitrary", {R11,R12,R22}=Rlist; found=True];
W={{1/a,0},{0,1/b}}; Rmat={{R11,R12},{R12,R22}};
WRW=Transpose[W].Rmat.W; Return[{WRW,found}]]

Figure 7. A Mathematica implementation of the rectangular panel template (31).

\[
\begin{bmatrix}
1 & 0 & y_1 & x_1 & 0 & y_1 \\
0 & 1 & -x_1 & 0 & y_1 & x_1 \\
1 & 0 & y_2 & x_2 & 0 & y_2 \\
0 & 1 & -x_2 & 0 & y_2 & x_2 \\
1 & 0 & y_3 & x_3 & 0 & y_3 \\
0 & 1 & -x_3 & 0 & y_3 & x_3 \\
1 & 0 & y_4 & x_4 & 0 & y_4 \\
0 & 1 & -x_4 & 0 & y_4 & x_4
\end{bmatrix}
= \frac{1}{3}
\begin{bmatrix}
2 & 0 & -b & -a & 0 & -b \\
0 & 2 & a & 0 & -b & -a \\
2 & 0 & -b & a & 0 & b \\
0 & 2 & -a & 0 & -b & a \\
2 & 0 & a & 0 & b & 0 \\
0 & 2 & a & 0 & b & a \\
2 & 0 & b & -a & 0 & b \\
0 & 2 & a & 0 & b & -a
\end{bmatrix}
\]

The six columns of \(G_{rc}\) span the rigid body modes and constant strain states evaluated
at the nodes (these bases are not orthonormalized as that property is not required here). It
is readily checked that \(H_h G_{rc} = 0\). Therefore those modes, and any linear combination
thereof, are orthogonal to the higher order stiffness: \(K_h G_{rc} = 0\). So the role of \(H_h\) is
essentially that of a geometric projector.

A Mathematica implementation of (31) as module RectPanel4TemplateStiffness
is shown in Figure 7. The module arguments are the rectangle dimensions as list \{a,b\}, the
elasticity matrix as list Emat=[{E11,E12,E13},{E12,E22,E23},{E13,E23,E33}],
the compliance matrix as Cmat=[{C11,C12,C13},{C12,C22,C23},{C13,C23,C33}],
the thickness h, the name as one of "Stress", "Strain", "Disp", "Q6", "QM6" or
"Arbitrary", and finally the list Rlist=[R11,R12,R22]. The latter is used if the name
is "Arbitrary". This comes handy for finding the signature of known elements leaving
the entries of Rlist symbolic and using the Solve command to match existing or new
Table 1. A Clone Gallery

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Clones and sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>StressRP (a.k.a. BORP)</td>
<td>5-stress-mode element of Section 4</td>
<td>Direct derivation: TCMT$^1$, Gallagher$^8$ Pian 5-mode stress hybrid$^{27,26,29}$ Wilson-Taylor-Doherty-Ghaboussi Q$^6$$^8$ Taylor-Wilson-Beresford QM$^6$$^6$ Belytschko-Liu-Engelmann QBI$^7$$^0$ SRI of iso-P with $E$ split as per (54)</td>
</tr>
<tr>
<td>StrainRP</td>
<td>5-strain-mode element of Section 5</td>
<td>MacNeal QUAD$^4$$^3$$^0$,7$^1$ SRI of iso-P with $E$ split as per (56)</td>
</tr>
<tr>
<td>DispRP</td>
<td>Bilinear iso-P element of Section 6</td>
<td>Argyris$^5$ as edge stiffened rectangular panel Taig-Kerr$^6$$^7$ as specialized quadrilateral</td>
</tr>
</tbody>
</table>

Note 1: Many plane stress models listed above were derived for quadrilateral geometries, and a few as membrane component of shells. The right-hand-column classification only pertains to the rectangular panel specialization. For example, Q$^6$ and QM$^6$ differ for non-parallelogram shapes.

Note 2: Instances of the stress-hybrid and displacement-bubble-function “futile families” studied in Section 11 are omitted, as they lack practical value.

Note 3: Post-1990 clones (e.g. EAS$^{54}$) omitted to save space. See Lautersztajn and Samuelsson$^7$$^2$ for a recent survey.

7.3 Requirements

An acceptable template fulfills four conditions: (C) consistency, (S) stability (correct rank), (I) observer invariance and (P) parametrization. These are discussed at length in other papers.$^{73-79}$ Conditions (C) and (S) are imposed to ensure convergence as the mesh size is reduced by enforcing a priori satisfaction of the Individual Element Test (IET) of Bergan and Hanssen.$^{56,57}$ Condition (P) means that the template contains free parameters or free matrix entries. In the case of (31), the simplest choice of parameters are the entries $R_{11}$, $R_{12}$, $R_{22}$ themselves. To fulfill stability, $R_{11} > 0$, $R_{22} > 0$ and $R_{11}R_{22} - R_{12}^2 > 0$. Parametrization facilitates performance optimization as well as tuning elements, or combinations of elements, to fulfill specific needs.

Using the IET as departure point it is not difficult to show$^{80}$ that (31), under the stated restrictions on $R$, includes all stiffnesses that satisfy the IET and stability. Observer invariance is a moot point for this element since $\{x, y\}$ are side aligned. Thus (31) is in fact a universal template for the rectangular panel.

7.4 Instances, Signatures, Clones

Setting the free parameters to specific values yields element instances. The set of free parameters is called the template signature, a term introduced in previous papers.$^{77,78}$ Borrowing terminology from biogenetics, the signature may be viewed as an “element DNA” that uniquely characterizes it as an individual entity. Elements derived by different techniques that share the same signature are called clones.
One of the “template services” is automatic identification of clones. If two elements fitting the template (31) share $R_{11}$, $R_{12}$ and $R_{22}$, they are clones. Inasmuch as most FEM formulation schemes have been tried on the rectangular panel, it should come as no surprise that there are many clones, particularly of the stress element. Those published before 1990 are collected in Table 1. For example, the incompatible mode element Q6 of Wilson et al.68 is a clone of StressRP. The version QM6 of Taylor et al.69 which passes the patch test for arbitrary geometries, reduces to Q6 for rectangular and parallelogram shapes. Even for this simple geometry recognition of some of the coalescences took some time, as recently narrated by Pian.29

8 FINDING THE BEST

An universal template is nice to have. The obvious question arises: among the infinity of elements that it can generate, is there a best one? By construction all instances verify exactly the IET for rigid body modes and uniform strain states. Hence the optimality criterion must rely on higher order patch tests.

8.1 The Bending Tests

The obvious tests involve response to in-plane bending along the side directions. This leads to comparisons in the form of energy ratios. These have been used since 1984 to tune up the higher order stiffness of triangular elements.58–62,81 An extension introduced in this article is consideration of arbitrary anisotropic material. All symbolic calculations were carried out with Mathematica.

The $x$ bending test is depicted in Figure 8. A Bernoulli-Euler plane beam of thin rectangular cross-section with height $b$ and thickness $h$ (normal to the plane of the figure) is bent under applied end moments $M_x$. The beam is fabricated of anisotropic material with the stress-strain law $\sigma = E\varepsilon$ of (2)2. Except for possible end effects the exact solution of the beam problem (from both the theory-of-elasticity and beam-theory standpoints) is a constant bending moment $M(x) = M_x$ along the span. The associated stress field is $\sigma_{xx} = -M_x \gamma/b$, $\sigma_{yy} = \sigma_{xy} = 0$, where $I_b = \frac{1}{12}bh^3$.

For the $y$ bending test, depicted in Figure 9, the beam cross section has height $a$ and thickness $h$, and is subjected to end moments $M_y$. The exact solution is $M(y) = M_y$. The associated stress field is $\sigma_{yy} = M_y \gamma/a$, $\sigma_{xx} = \sigma_{xy} = 0$, where $I_a = \frac{1}{12}ha^3$. For comparing with the FEM discretizations below, the internal (complementary) energies taken up by beam segments of lengths $a$ and $b$ in the configurations of Figures 8 and 9, respectively, are

$$U_x^{\text{beam}} = \frac{6aC_{11}M_x^2}{b^3h}, \quad U_y^{\text{beam}} = \frac{6bC_{22}M_y^2}{a^3h} \quad (36)$$
For the 2D element tests, each beam is modeled with one layer of identical 4-node rectangular panels dimensioned \( a \times b \) as shown in Figures 8 and 9. The aspect ratio \( b/a \) is denoted by \( \gamma \). By analogy with the exact solution, all rectangles in the finite element model will undergo the same deformations and stresses. We can therefore consider a typical element. For \( x \)-bending the exact stress distribution is represented by (7) on taking \( \mu_4 = -M_x b/I_b = -12M_x/(b^2 h) \) and \( \mu_1 = \mu_2 = \mu_3 = \mu_5 = 0 \). The rigid body mode amplitudes are chosen to be zero for convenience: \( \mu_6 = \mu_7 = \mu_8 = 0 \). Inserting these \( \mu_i \) into (14) we get the node displacement vector

\[
\mathbf{u}_{bx} = \frac{12M_x C_{11}a}{b^2 h} \left[ -1 \ 0 \ 1 \ 0 \ -1 \ 0 \ 1 \ 0 \right]^T.
\]

(37)

Likewise, for the \( y \)-bending test the element stress field is obtained by taking \( \mu_5 = M_y a/I_a = 12M_y/(a^2 h) \) and \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = \mu_7 = \mu_8 = 0 \). The node displacement vector given by (14) is

\[
\mathbf{u}_{by} = \frac{12M_y C_{22}b}{a^2 h} \left[ 0 \ 1 \ 0 \ -1 \ 0 \ 1 \ 0 \ -1 \right]^T.
\]

(38)

The strain energies absorbed by the panel element under these applied node displacements are \( U_x^{\text{panel}} = \frac{1}{2} \mathbf{u}_{bx}^T \mathbf{K}_{bx} \mathbf{u}_{bx} \) and \( U_y^{\text{panel}} = \frac{1}{2} \mathbf{u}_{by}^T \mathbf{K}_{by} \mathbf{u}_{by} \), respectively. Define the bending energy ratios as

\[
\frac{r_x}{U_x^{\text{beam}}} = \frac{U_x^{\text{panel}}}{U_x^{\text{beam}}}, \quad \frac{r_y}{U_y^{\text{beam}}} = \frac{U_y^{\text{panel}}}{U_y^{\text{beam}}}. \quad (39)
\]

These happen to be the ratios of the exact (beam) displacement solution to that of the rectangular panel solution. Hence \( r_x = 1 \) means that we get the exact answer under \( M_x \), that is, the panel is \( x \)-bending exact. If \( r_x > 1 \) or \( r_x < 1 \) the panel is overstiff or overflexible in \( x \) bending, respectively. Likewise for \( y \)-bending, \( r_y = 1 \) for any aspect ratio \( \gamma = b/a \) and arbitrary material properties the element is called \( \text{bending optimal} \). If \( r_x >> 1 \) if \( a >> b \) and/or \( r_y >> 1 \) if \( a << b \) the element is said to experience \( \text{aspect ratio locking} \) along the \( x \) or \( y \) direction, respectively. This is called \( \text{shear locking} \) in the FEM literature because it is traceable to spurious shear energy, as shown in Section 8.4.

8.2 The Optimal Panel

Applying the tests to the template (31) yields

\[
\frac{r_x}{U_x^{\text{beam}}} = 3C_{11} R_{11}, \quad \frac{r_y}{U_y^{\text{beam}}} = 3C_{22} R_{22}.
\]

(40)

Clearly to get \( r_x = r_y = 1 \) for any aspect ratio we must take

\[
R_{11} = \frac{1}{3} C_{11}^{-1}, \quad R_{22} = \frac{1}{3} C_{22}^{-1}
\]

(41)

Since \( R_{12} \) is absent from (41) one can set \( R_{12} = 0 \) for convenience. Comparing to the \( \mathbf{R}_\sigma \) of (33) shows that the 5-parameter stress model of TCMT and its clones is the \text{bending-optimal rectangular panel} \( \). For isotropic material \( R_{11} = R_{22} = \frac{1}{3} E \). The StressRP template instance will be henceforth also identified by the acronym BORP, for Bending Optimal Rectangular Panel.
8.3 The Strain Element Does Not Lock

It is interesting to apply the result (40) to other template instances. The StrainRP element generated by the $R_e$ of (33) gives

$$r_x = C_{11}E_{11}, \quad r_y = C_{22}E_{22}. \quad (42)$$

If the material is isotropic, $C_{11} = C_{22} = 1/E$ and $E_{11} = E_{22} = E/(1 - \nu^2)$. This yields $r_x = r_y = 1/(1 - \nu^2)$, which varies between 1 and 4/3. For an orthotropic body with principal material axes aligned with the rectangle sides, $E_{11} = E_1/(1 - \nu_{12}v_{21})$, $E_{22} = E_2/(1 - \nu_{12}v_{21})$, $C_{11} = 1/E_1$, $C_{22} = 1/E_2$, and $r_x = r_y = 1/(1 - \nu_{12}v_{21})$. These are independent of the aspect ratio $\nu$. Consequently StrainRP and its clones do not lock, although the element is not generally optimal. Note that if $C_{11}E_{11}$ and/or $C_{22}E_{22}$ differ widely from 1, as may happen in highly anisotropic materials, the bending performance will be poor. The example of Section 12.2 displays this vividly.

8.4 But the Displacement Element Does

Instance DispRP is generated by the $R_u$ of (33). Inserting its entries into (40) we get

$$r_x = C_{11}(E_{11} + E_{33}\nu^2) = \frac{(E_{22}E_{33} - E_{23}^2)(E_{11} + E_{33}\nu^2)}{\det(E)}, \quad r_y = C_{22}(E_{22} + E_{33}\nu^2) = \frac{(E_{11}E_{33} - E_{13}^2)(E_{22} + E_{33}\nu^2)}{\det(E)}. \quad (43)$$

in which $\det(E) = E_{11}E_{22}E_{33} + 2E_{12}E_{13}E_{23} - E_{11}E_{23}^2 - E_{22}E_{13}^2 - E_{33}E_{12}^2$. For an isotropic material

$$r_x = \frac{2 + \nu^2(1 - \nu)}{2(1 - \nu^2)}, \quad r_y = \frac{1 + 2\nu^2 - \nu}{2\nu^2(1 - \nu^2)}. \quad (44)$$

These relations clearly indicate aspect ratio locking for bending along the longest side dimension. For instance if $\nu = 0$ and $a = 10b$, whence $\nu = a/b = 10$, then $r_x = 51$ and DispRP is over 50 times stiffer in $x$ bending than the Bernoulli-Euler beam element. The expression (43) makes clear that locking happens for any material law as long as $E_{33} \neq 0$. Since this is the shear modulus, the name shear locking used in the FEM literature is justified.

8.5 Multiple Element Layers

Results of the energy bending test can be readily extended to predict the behavior of $2n$ ($n = 1, 2, \ldots$) identical layers of elements symmetrically placed through the beam height. If $2n$ layers are placed along the $y$ direction in the configuration of Figure 8 and $\gamma$ stays the same, the energy ratio becomes

$$r_x^{(2n)} = \frac{2n - 1 + r_x}{2^n}, \quad (45)$$

where $r_x$ is the ratio (40) for one layer. If $r_x \equiv 1$, $r_x^{2n} \equiv 1$ so bending exactness is maintained, as can be expected. For example, if $n = 1$ (two element layers), $r_x^{(2)} = (3 + r_x)/4$. The same result holds for $r_y$ if $2n$ layers are placed along the $x$ direction in the configuration of Figure 9.
MORPHING TO BEAM-COLUMN

Morphing means transforming an individual element or macroelement into a simpler model using kinematic constraints. Often the simpler element has lower dimensionality. For example a plate bending macroelement may be morphed to a Bernoulli-Euler beam or to a torqued shaft. To illustrate the idea consider morphing the rectangular panel of Figure 10 into the two-node beam-column element shown on the right of that Figure. The length, cross sectional area and moment of inertia of the beam-column element, respectively, are denoted by \( L = a \), \( A = bh \) and \( I_{zz} = \frac{b^3 h}{12} = \frac{a^3 h}{(12 \gamma^3)} \), respectively.

The transformation between the freedoms of the panel and those of the beam-column is

\[
\mathbf{u}_R = \begin{bmatrix}
   u_{x1} \\
   u_{y1} \\
   u_{x2} \\
   u_{y2} \\
   u_{x3} \\
   u_{y3} \\
   u_{x4} \\
   u_{y4}
\end{bmatrix} = \begin{bmatrix}
   1 & 0 & \frac{1}{2}b & 0 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & \frac{1}{2}b & 0 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
   \bar{u}_{x1} \\
   \bar{u}_{y1} \\
   \bar{\theta}_1 \\
   \bar{u}_{x2} \\
   \bar{u}_{y2} \\
   \bar{\theta}_2
\end{bmatrix} = \mathbf{T}_m \mathbf{\bar{u}}_m. \tag{46}
\]

where a superposed bar distinguishes the beam-column freedoms grouped in array \( \mathbf{\bar{u}}_m \). As source select StressRP fabricated of isotropic material. The morphed beam-column element stiffness is

\[
\mathbf{K}_m = \mathbf{T}_m^T \mathbf{K}_\sigma \mathbf{T}_m = \frac{E}{L} \begin{bmatrix}
   A & 0 & 0 & -A & 0 & 0 & 0 & 0 \\
   0 & 12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & -12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & 0 \\
   0 & 6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & -6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & 0 \\
   -A & 0 & 0 & A & 0 & 0 & 0 & 0 \\
   0 & 12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & -12c_{22}I_{zz}/L^2 & 6c_{23}I_{zz}/L & 0 & 0 \\
   0 & 6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & -6c_{23}I_{zz}/L & 4c_{33}I_{zz} & 0 & 0
\end{bmatrix}, \tag{47}
\]

in which \( c_{22} = c_{23} = \frac{1}{4} \gamma^2/(1 + \nu) \), and \( c_{33} = \frac{1}{4}(1 + 3c_{22}) \). The entries in rows/columns 1 and 4 form the well known two-node bar stiffness. Those in rows and columns 2, 3, 5 and 6 are dimensionally homogeneous to those of a plane beam, and may be grouped into the following matrix configuration:

\[
\mathbf{K}_m^{beam} = \frac{E I_{zz}}{L} \begin{bmatrix}
   1 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & -1 & 0 \\
   0 & 0 & 0 & 0 & 0 \\
   0 & -1 & 0 & 1 & 0
\end{bmatrix} + \beta_m \begin{bmatrix}
   12/L^2 & 6/L & -12/L^2 & 6/L & 6/L \\
   6/L & 3 & -6/L & 3 & 6/L \\
   -12/L^2 & -6/L & 12/L^2 & -6/L & 6/L \\
   6/L & 3 & -6/L & 3 & 6/L
\end{bmatrix}. \tag{48}
\]
where $\beta_m = c_{22} = c_{23} = \frac{1}{2} \gamma^2/(1 + \nu)$. But (48), with $\beta_m$ replaced by a free parameter $\beta$, happens to be the universal template of a prismatic plane beam. Mass-stiffness template combinations have been studied for dynamics and vibration using Fourier methods.

The basic stiffness on the left characterizes the pure-bending symmetric response to a uniform moment, whereas the higher-order stiffness on the right characterizes the antisymmetric response to a linearly-varying, bending moment of zero mean. For the Bernoulli-Euler beam constructed with cubic shape functions, $\beta = 1$. For the Timoshenko beam, the exact equilibrium model developed in §5.6 of Przemieniecki is matched by $\beta = \beta_{C0} = 1/(1 + \phi)$, in which $A_s = 5bh/6$ is the shear area and $G = \frac{1}{2} E/(1 + \nu)$ the shear modulus. The morphed $\beta_m$ is always higher than $\beta_{C0}$ for all $0 \leq \nu \leq \frac{1}{2}$ and aspect ratios $\gamma > 0$. This indicates that in beam-like problems involving transverse shear the rectangular panel will be stiffer than the exact $C^0$ beam model. For example if $\nu = 1/4$

$$\frac{\beta_{C0}}{\beta_m} = \frac{5}{2(3 + \gamma^2)}. \quad (49)$$

This never exceeds $5/6$ and goes to zero as $\gamma \to \infty$. This reflects the fact that a 4-node panel can only respond to such antisymmetric nodal motions by deforming in pure shear. The symmetric response, however, is exact for any aspect ratio $\gamma$, confirming the optimality of StressRP (= BORP). Observe also that what was a higher order patch test on the two-triangle mesh unit becomes a basic (constant-moment) patch test on the morphed element. This is typical of morphing transformations that reduce spatial dimensionality.

For nonoptimal elements, one finds that the basic stiffness of the morphed beam is wrong except under very special circumstances. For example isotropic StrainRP with zero $\nu$, or one of the SRI elements studied next.

10 A G3 DEVICE: SELECTIVE REDUCED INTEGRATION

The three canonical models of Sections 4-6 were known by the end of Generation 2. Next a third generation tool will be studied in the context of templates. Full Reduced Integration (FRI) and Selective Reduced Integration (SRI) emerged during 1969–72 as tools to “unlock” isoparametric displacement models. Initially labeled as “variational crimes” by Strang they were eventually justified through lawful association with mixed variational methods. Both FRI and SRI turned out to be particularly useful for legacy and nonlinear codes because they allow shape function and numerical integration modules to be reused. For the 4-node panel only SRI is considered because FRI leads to rank deficiency: $R_{11} = R_{12} = R_{22} = 0$. Two questions can be posed:

(i) Can the template (31)–(32) be reproduced for any material law by a SRI scheme?
(ii) Can BORP be cloned for any material law by a SRI scheme that is independent of the aspect ratio?

As shown below, the answers are (i): yes if $R_{12} = 0$; (ii): yes.

10.1 Concept and Notation

In the FEM literature, SRI identifies a scheme for forming $K$ as the sum of two or more matrices computed with different integration rules and different constitutive properties, within the framework of the isoparametric (iso-P) displacement model. We will focus here on a two-way constitutive decomposition. Split the plane stress constitutive matrix $E$ into

$$E = E_I + E_{II}. \quad (50)$$

The iso-P displacement formulation leads to the expression $K = \int_{A^e} h B_u^T E B_u \, d\Omega$ where $A^e$ is the element area and $B_u$ the iso-P strain-displacement matrix. To apply SRI
isoparametric model is the 4-node bilinear element. Rules (I) and (II) will be the 1 rule (I) for the 
(one point) and 2
elasticity matrix is

$$K = \int_{A'} h \mathbf{B}_i^T \mathbf{E}_i \mathbf{B}_i \, d\Omega + \int_{A'} h \mathbf{B}_i^T \mathbf{E}_{II} \mathbf{B}_i \, d\Omega = K_I + K_{II}. \quad (51)$$

The two matrices in (51) are done through different numerical quadrature schemes:
rule (I) for the first integral and rule (II) for the second. For the rectangular panel the isoparametric model is the 4-node bilinear element. Rules (I) and (II) will be the $1 \times 1$ (one point) and $2 \times 2$ (4-point) Gauss product rules, respectively. A general split of the elasticity matrix is

$$\mathbf{E} = \mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} E_{11}(1 - \rho_1) & E_{12}(1 - \rho_3) & E_{13}(1 - \tau_2) \\ E_{12}(1 - \rho_3) & E_{22}(1 - \rho_2) & E_{23}(1 - \tau_3) \\ E_{13}(1 - \tau_2) & E_{23}(1 - \tau_3) & E_{33}(1 - \tau_3) \end{bmatrix}, \quad (52)$$

in which $\rho_1, \rho_2, \rho_3, \tau_1, \tau_2,$ and $\tau_3$ are dimensionless coefficients to be chosen.

10.2 The Case $R_{12} = 0$

A template with $R_{12} = 0$ and arbitrary $\{R_{11}, R_{22}\}$ is matched by taking

$$\rho_1 = \frac{1 - 3 R_{11}}{E_{11}}, \quad \rho_2 = \frac{1 - 3 R_{22}}{E_{22}}, \quad \tau_1 = \tau_2 = \tau_3 = 1. \quad (53)$$

Since $\rho_3$ does not appear, it is convenient to set it to one to get a diagonal $\mathbf{E}_{II}$. The resulting split is

$$\mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} E_{11} - 3 R_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} - 3 R_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} 3 R_{11} & 0 & 0 \\ 0 & 3 R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (54)$$

To get the optimal element (BORP) set $R_{11} = \frac{1}{3} C_{11}^{-1}$ and $R_{22} = \frac{1}{3} C_{22}^{-1}$:

$$\mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} E_{11} - C_{11}^{-1} & E_{12} & E_{13} \\ E_{12} & E_{22} - C_{22}^{-1} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} C_{11}^{-1} & 0 & 0 \\ 0 & C_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (55)$$

For isotropic material this becomes

$$\mathbf{E}_I + \mathbf{E}_{II} = \frac{E}{1 - \nu^2} \begin{bmatrix} \nu^2 & \nu & 0 \\ \nu & \nu^2 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (56)$$

To match the (suboptimal) StrainRP, in which $R_{11} = \frac{1}{3} E_{11}$ and $R_{22} = \frac{1}{3} E_{22}$ the appropriate split is

$$\mathbf{E}_I + \mathbf{E}_{II} = \begin{bmatrix} 0 & E_{12} & E_{13} \\ E_{12} & 0 & E_{23} \\ E_{13} & E_{23} & 0 \end{bmatrix} + \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (57)$$
For isotropic material this becomes

\[
E_1 + E_{II} = E \begin{bmatrix} 0 & \nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} + E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\] (58)

Some FEM books suggest using the dilatational (a.k.a. volumetric, bulk) elasticity law for \(E_1\). As can be seen, the recommendation is incorrect for this element.

10.3 The Case \(R_{12} \neq 0\)

The case \(R_{12} \neq 0\) arises in anisotropic displacement models in which \(E_{13} \neq 0\) and/or \(E_{23} \neq 0\). Now \(\tau_2\) and \(\tau_3\) must verify \(E_{13} \gamma^{-1} \tau_2 + E_{23} \gamma \tau_3 = E_{13} \gamma^{-1} + E_{23} \gamma - 3R_{12}\). Solve for that \(\gamma\) (\(i = 2, 3\)) that has an associated nonzero modulus. Note that the aspect ratio \(\gamma\) will generally appear in the SRI rule.

This case lacks practical interest because optimality can be achieved with \(R_{12} = 0\). But for DispRP an obvious solution that eliminates all aspect ratio dependent is \(\rho_1 = \rho_2 = \rho_3 = \tau_1 = \tau_2 = \tau_3 = 0\), whence \(E_1 = 0, E_{II} = E\) and the fully integrated isoP element, which locks, is recovered.

10.4 Selective Directional Integration

The template can also be generated by non-Gaussian rules. For example, the following three-way directional split

\[
E_1 + E_{II} + E_{III} = \begin{bmatrix} E_{11} - C_{11}^{-1} & E_{12} & E_{13} \\ E_{12} & E_{22} - C_{22}^{-1} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} + \begin{bmatrix} C_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\] (59)

generates the optimal panel in conjunction with three rules. Rule (I) is one-point Gauss with \(\{\xi, \eta\} = \{0, 0\}\) and weight 4; Rule (II) has two points on the \(y = 0\) median: \(\{\xi, \eta\} = \{0, \pm 1/\sqrt{3}\}\) with weight 2; rule (III) has two points on the \(x = 0\) median: \(\{\xi, \eta\} = \{\pm 1/\sqrt{3}, 0\}\) with weight 2. This selective directional integration is difficult to extend to arbitrary quadrilaterals while preserving observer invariance.

11 FUTILE FAMILIES

Families are template subsets that arise naturally from specific methods as function of discrete or continuous decision parameters. To render the concept more concrete two historically important, albeit practically useless, families for the rectangular panel are considered next.

11.1 Equilibrium Stress Hybrids

This family was studied in the late 1960s by hapless authors with the not unreasonable belief that “more is better.” It is obtained by generalizing the 5-parameter stress form of Section 4 with a polynomial series in \(\{x, y\}\). An obvious choice is to make \(\sigma_{xx}, \sigma_{yy}\) and \(\sigma_{xy}\) complete polynomials in \(\{x, y\}\):

\[
\sigma_{xx} = \sum_{i,j} a_{ij} x^i y^j, \quad \sigma_{yy} = \sum_{i,j} b_{ij} x^i y^j, \quad \sigma_{xy} = \sum_{i,j} c_{ij} x^i y^j, \quad i \geq 0, \quad j \geq 0, \quad i + j \leq n.
\] (60)

For a complete expansion of order \(n \geq 0\) one gets \(3(n+1)(n+2)/2\) coefficients. Imposing strongly the two internal equilibrium equations (1) for zero body forces reduces the set to \(n_\sigma = 3 + 3n + n^2\) independent coefficients. For \(n = 0, 1, 3, 5\) and 7 this gives \(n_\sigma = 3, 7, 13, 21\) and 31 coefficients, respectively. (Only odd \(n\) is of interest beyond
Figure 12. Representation of template families on the \([r_x, r_y]\) plane.

\(n = 0\), since terms with \(i + j = 2, 4, \ldots\) etc., cancel out on integrating strains over the rectangle and have no effect on the element stiffness.)

The stiffness equations of this family can be obtained by the hybrid stress method of Pian and Tong.\(^{28,49}\) To display the effect of \(n_\sigma\), the signature of the template (31)–(32) and the associated bending energy ratios were calculated for aspect ratio \(\gamma = a/b = 4\), isotropic material with modulus \(E\) and Poisson’s ratio \(\nu = 1/3\).

Table 2. Signatures and Bending Ratios for Stress Hybrid Family

<table>
<thead>
<tr>
<th>(n_\sigma)</th>
<th>5</th>
<th>7</th>
<th>13</th>
<th>21</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_{11}/E)</td>
<td>0.33333</td>
<td>2.21173</td>
<td>2.21762</td>
<td>2.22125</td>
<td>2.22235</td>
</tr>
<tr>
<td>(R_{22}/E)</td>
<td>0.33333</td>
<td>0.35650</td>
<td>0.35967</td>
<td>0.35979</td>
<td>0.35981</td>
</tr>
<tr>
<td>(R_{12})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(r_x)</td>
<td>1.00000</td>
<td>6.63518</td>
<td>6.65386</td>
<td>6.66375</td>
<td>6.66705</td>
</tr>
<tr>
<td>(r_y)</td>
<td>1.00000</td>
<td>1.06949</td>
<td>1.07900</td>
<td>1.07938</td>
<td>1.07944</td>
</tr>
</tbody>
</table>

The results are collected in Table 2. The bending energy ratios are displayed in Figure 12. Increasing the number of stress terms rapidly stiffens the element in \(x\)-bending. This is an instance of what may be called \textit{equilibrium stress futility}: adding more stress terms makes things worse. (The phenomenon is well known but a representation such as that in Figure 12 is new.) As \(n_\sigma \to \infty\) the template signature approaches the limit \(R_{11}/E \approx 0.2224\) and \(R_{22}/E \approx 0.3599\) to 4 places.

11.2 Bubble-Augmented Isoparametrics

A second family can be generated by starting from the conforming iso-P element DispRP of Section 6, and injecting \(n_b\) displacement bubble functions. [Bubble are shape functions that vanish over the element boundaries.] The idea is also a late-G2 curiosity but has resurfaced recently. Results for 2 and 18 bubbles (associated with 1 and 9 internal nodes, respectively) are collected in Table 3 and displayed also in Figure 12.

As can be expected injecting bubbles makes the element more flexible but the improvement is marginal. If \(n_b \to \infty\) the signature approaches that of the \(n_\sigma \to \infty\) hybrid-stress model of the previous subsection. For all this extra work (these models rapidly become expensive on account of high order Gauss integration rules and DOF condensation), \(r_x\) decreases from 7.12 to 6.67. This is a convincing illustration of \textit{bubble futility}.
Table 3. Signatures and Bending Ratios for Bubble-Augmented Family

<table>
<thead>
<tr>
<th>n_b</th>
<th>0</th>
<th>2</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>R_{11}/E</td>
<td>2.37501</td>
<td>2.23894</td>
<td>2.22546</td>
</tr>
<tr>
<td>R_{22}/E</td>
<td>0.38281</td>
<td>0.36088</td>
<td>0.35998</td>
</tr>
<tr>
<td>R_{12}</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>r_x</td>
<td>7.12505</td>
<td>6.71683</td>
<td>6.67637</td>
</tr>
<tr>
<td>r_y</td>
<td>1.14844</td>
<td>1.08265</td>
<td>1.07994</td>
</tr>
</tbody>
</table>

Figure 13. Slender cantilever beam for Examples 1 and 2. A 16 × 1 FEM mesh with γ = 1 is shown in (b).

Figure 12 also marks the energy ratios of the StrainRP element. For this instance 
R_{11}/E = R_{22}/E = 3/8 = 0.375 and r_x = r_y = 1.125. Consequently the element is only slightly overstiff. Increasing the number of strain terms, however, would lead to another “futile family.”

12 NUMERICAL EXAMPLES

Three benchmark examples involving cantilever beams are studied below. The sequel presents benchmarks involving general quadrilateral shapes and thin-wall shell structures.

12.1 Example 1: Slender Isotropic Cantilever

The slender 16:1 cantilever beam of Figure 13(a) is fabricated of isotropic material, with E = 7680, ν = 1/4 and G = (2/5)E = 3072. The dimensions are shown in the Figure. Two end load cases are considered: an end moment M = 1000 and a transverse end shear P = 48000/1027 = 46.7381. Both tip deflections δ_C = u_{yC} from beam theory: ML^2/(2EI_z) and PL^3/(3EI_z) + PL/(GA_z), in which I_z = b^3h/12 and A_z = 5A/6 = 5bh/6, are exactly 100. For the second load case the shear deflection is only 0.293% of u_{yC}; thus the particular expression used for A_z is not very important.

Regular meshes with only one element (N_z = 1) through the beam height are considered. The number N_x of elements along the span is varied from 1 to 64, giving elements with aspect ratios from γ = 1 through γ = 1/4. The root clamping condition is imposed by setting u_x to zero at both root nodes, but u_y is only fixed at the lower one thus allowing for Poisson’s contraction at the root.

Tables 4 and 5 report computed tip deflections u_{yC} for several element types. The first three rows list results for the 3 rectangular panel models of Sections 4–6. The last three rows give results for selected triangular elements. BODT is the Bending Optimal Drilling Triangle: a 3-node membrane element with drilling freedoms studied in previous papers. ALL-EX is the exactly integrated 1988 Allman triangle with drilling freedoms. CST is the Constant Strain Triangle, also called linear triangle and Turner triangle. Both ALL-EX and BODT have three freedoms per node whereas all others
have two. To get exactly 100.00% from BODT under an end-moment requires particular attention to the end load consistent lumping.\textsuperscript{92}

BOMP is exact for all $\gamma$ under end-moment and converges rapidly under end-shear. The performance of BODT is similar, inasmuch as this triangle is constructed to be bending exact in rectangular-mesh units. (In the end-shear load case BORP and BODT, which morph to different beam templates, converge to slightly different limits as $\gamma \rightarrow 0$.) StrainRP is about 6% stiffer than BORP, which can be expected since $1/(1 - \nu^2) = 16/15$. DispRP, as well as the triangles ALL-EX and CST, lock as $\gamma$ increases.

The response for more element layers through the height can be readily estimated from equation (45). Consequently those results are omitted to save space. For example, to predict the DispRP answer on a $8 \times 4$ mesh under end-moment, proceed as follows. The aspect ratio is $\gamma = 8$. From the $\gamma = 8$ column of Table 4 read off $r_x = 100/3.75 = 26.667$. Set $n = 2$ in (45) to get $r_x^{(4)} = (15 + r_x)/16 = 2.60417$. The estimated tip deflection is $100/2.60417 = 38.40$. Running the program gives $\delta_C = 38.3913$ as average of the $y$ displacement of the two end nodes. Predictions for the end-shear-load case will be less accurate but sufficient for quick estimation.

12.2 Example 2: Slender Anisotropic Cantilever

Next assume that the beam of Figure 13(a) is fabricated of anisotropic material with the elasticity properties

$$E = \begin{bmatrix} 880 & 600 & 250 \\ 600 & 420 & 150 \\ 250 & 150 & 480 \end{bmatrix}, \quad C = E^{-1} = \frac{1}{35580} \begin{bmatrix} 1791 & -2505 & -150 \\ -2505 & 3599 & 180 \\ -150 & 180 & 96 \end{bmatrix}. \quad (61)$$

That these are physically realizable can be checked by getting the eigenvalues of $E$: $\{1386.1, 387.3, 6.63\}$, whence both $E$ and $C$ are positive definite. The load magnitudes are adjusted to get beam-theory tip deflections of 100: $M = 2.58672$ and $P = 0.121153$. Since

$$E_{11}C_{11} = 44.297 \quad (62)$$

the energy ratio analysis of Sections 8.3–8.4, through equations (42) and (43), predicts that the strain and displacement models will be big losers, because $r_x \geq 44.297$. This is verified in Tables 6 and 7, which report computed tip deflections $u_{yC}$ for the three rectangular panel models. While BOMP shines, the strain and displacement models are way off, regardless of how many elements one puts along $x$.

Putting more element layers through the height will help StrainRP and DispRP but too slowly to be practical. To give an example, a $128 \times 8$ mesh of StrainRP (or clones) under end moment will have $r_x^{(8)} = (63 + 44.297)/64 = 1.676$ and estimated deflection of $100/1.676 = 59.647$. Running that mesh gives $u_{yC} = 59.65$. So using over 2000 freedoms in this fairly trivial problem the results are still off by about 40%.

12.3 Example 3: Short Cantilever Under End Shear

The shear-loaded cantilever beam defined in Figure 14 has been selected as a test problem for plane stress elements by many investigators since proposed in the writer’s thesis.\textsuperscript{94} A full root-clamping condition is implemented by constraining both displacement components to zero at nodes located on at the root section $x = 0$. The applied shear load varies parabolically over the end section and is consistently lumped at the nodes. The main comparison value is the tip deflection $\delta_C = u_{yC}$ at the center of the end cross section. Reference\textsuperscript{81} recommends $\delta_C = 0.35601$, which is also adopted here. The converged value of digits 4-5 is clouded by the mild singularity developing at the root section. This singularity is displayed for $\sigma_{xy}$ in the form of an intensity contour plot in Figure 15.
Table 4  Tip Deflections (exact=100) for Slender Isotropic Cantilever under End Moment

<table>
<thead>
<tr>
<th>Element</th>
<th>Mesh: x-subdivisions × y-subdivisions (N_x × N_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 × 1 2 × 1 4 × 1 8 × 1 16 × 1 32 × 1 64 × 1</td>
</tr>
<tr>
<td></td>
<td>(γ = 16) (γ = 8) (γ = 4) (γ = 2) (γ = 1) (γ = 1/2) (γ = 1/4)</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>100.00 100.00 100.00 100.00 100.00 100.00 100.00</td>
</tr>
<tr>
<td>StrainRP</td>
<td>93.75 93.75 93.75 93.75 93.75 93.75 93.75</td>
</tr>
<tr>
<td>DispRP</td>
<td>0.97 3.75 13.39 37.49 68.18 85.71 91.60</td>
</tr>
<tr>
<td>ALL-EX</td>
<td>0.04 0.63 7.40 35.83 58.44 64.89 66.45</td>
</tr>
<tr>
<td>CST</td>
<td>0.32 1.25 4.46 12.50 22.73 28.57 30.53</td>
</tr>
<tr>
<td>BODT</td>
<td>100.00 100.00 100.00 100.00 100.00 100.00 100.00</td>
</tr>
</tbody>
</table>

Table 5  Tip Deflections (exact=100) for Slender Isotropic Cantilever under End Shear

<table>
<thead>
<tr>
<th>Element</th>
<th>Mesh: x-subdivisions × y-subdivisions (N_x × N_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 × 1 2 × 1 4 × 1 8 × 1 16 × 1 32 × 1 64 × 1</td>
</tr>
<tr>
<td></td>
<td>(γ = 16) (γ = 8) (γ = 4) (γ = 2) (γ = 1) (γ = 1/2) (γ = 1/4)</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>75.02 93.72 98.39 99.56 99.86 99.94 99.97</td>
</tr>
<tr>
<td>StrainRP</td>
<td>70.35 87.88 92.26 93.35 93.63 93.71 93.73</td>
</tr>
<tr>
<td>DispRP</td>
<td>0.24 0.69 6.36 35.18 59.59 65.70 67.03</td>
</tr>
<tr>
<td>ALL-EX</td>
<td>0.48 1.41 4.62 12.66 22.88 28.73 30.69</td>
</tr>
<tr>
<td>CST</td>
<td>100.00 100.00 100.00 100.00 100.00 100.00 100.00</td>
</tr>
</tbody>
</table>

Table 6  Tip Deflections (exact=100) for Slender Anisotropic Cantilever under End Moment

<table>
<thead>
<tr>
<th>Element</th>
<th>Mesh: x-subdivisions × y-subdivisions (N_x × N_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 × 1 2 × 1 4 × 1 8 × 1 16 × 1 32 × 1 64 × 1</td>
</tr>
<tr>
<td></td>
<td>(γ = 16) (γ = 8) (γ = 4) (γ = 2) (γ = 1) (γ = 1/2) (γ = 1/4)</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>100.00 100.00 100.00 100.00 100.00 100.00 100.00</td>
</tr>
<tr>
<td>StrainRP</td>
<td>2.26 2.26 2.26 2.26 2.26 2.26 2.26</td>
</tr>
<tr>
<td>DispRP</td>
<td>0.02 0.07 0.25 0.76 1.53 2.08 2.25</td>
</tr>
</tbody>
</table>

Table 7  Tip Deflections (exact=100) for Slender Anisotropic Cantilever under End Shear

<table>
<thead>
<tr>
<th>Element</th>
<th>Mesh: x-subdivisions × y-subdivisions (N_x × N_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 × 1 2 × 1 4 × 1 8 × 1 16 × 1 32 × 1 64 × 1</td>
</tr>
<tr>
<td></td>
<td>(γ = 16) (γ = 8) (γ = 4) (γ = 2) (γ = 1) (γ = 1/2) (γ = 1/4)</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>74.95 93.68 98.37 99.54 99.84 99.92 99.96</td>
</tr>
<tr>
<td>StrainRP</td>
<td>1.70 2.12 2.22 2.26 2.26 2.26 2.26</td>
</tr>
<tr>
<td>DispRP</td>
<td>0.02 0.07 0.25 0.75 1.52 2.06 2.23</td>
</tr>
</tbody>
</table>

Table 8 gives computed deflections for rectangular mesh units with aspect ratios of 1, 2 and 4, using the three canonical rectangular panel models and the three triangles identified in Example 1. For end deflection reporting the load was scaled by (100/0.35601) so that the “theoretical solution” becomes 100.00. (In comparing stress values the unscaled load of $P = 40$ was used.) There are no drastically small deflections because element aspect
Figure 14. Short cantilever under end-shear benchmark: $E = 30000$, $v = 1/4$, $h = 1$; root contraction not allowed, a $8 \times 2$ mesh is shown in (b).

Figure 15. Intensity contour plot of $\sigma_{xy}$ given by the $64 \times 16$ BORP mesh. Produced by Mathematica and Gaussian filtered by Adobe Photoshop. Stress node values averaged between adjacent elements. The root singularity pattern is clearly visible.

Figure 16. Distributions of $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{xy}$ at $x = 12$ given by the $64 \times 16$ BORP mesh. Stress node values averaged between adjacent elements. Note different stress scales. Deviations at $y = \pm 6$ (free edges) due to “upwinded” $y$ averaging.

ratios only go up to 4:1. Elements StressRP (BORP), StrainRP and BODT outperformed the others. There is little to choose between these 3 models, which is typical of isotropic materials. The BODT triangle is more versatile but carries one more freedom per node.

Figure 16 plots averaged node stress values at section $x = 12$ computed from the $64 \times 16$ BORP mesh. The agreement with the standard beam stress distribution (that section being sufficiently away from the root) is very good except for $\sigma_{xy}$ near the free edges $y = \pm 6$, at which the interelement averaging process becomes biased.

13 DISCUSSION AND CONCLUSIONS

What can templates contribute to FEM technology? Advantages in two areas are clear: Synthesis. Only one procedure (module, function, subroutine) is written to do many elements. This simplifies comparison and verification benchmarking, as well as streamlining
Table 8  Tip Deflections (exact = 100) for Short Cantilever under End Shear

<table>
<thead>
<tr>
<th>Element</th>
<th>Mesh: $x$-subdivisions $\times$ $y$-subdivisions ($N_x \times N_y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8 $\times$ 2</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>98.80</td>
</tr>
<tr>
<td>DispRP</td>
<td>88.83</td>
</tr>
<tr>
<td>ALL-EX</td>
<td>89.43</td>
</tr>
<tr>
<td>CST</td>
<td>55.09</td>
</tr>
<tr>
<td>BODT</td>
<td>101.68</td>
</tr>
<tr>
<td></td>
<td>4 $\times$ 2</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>97.22</td>
</tr>
<tr>
<td>StrainRP</td>
<td>95.67</td>
</tr>
<tr>
<td>DispRP</td>
<td>69.88</td>
</tr>
<tr>
<td>ALL-EX</td>
<td>70.71</td>
</tr>
<tr>
<td>CST</td>
<td>37.85</td>
</tr>
<tr>
<td>BODT</td>
<td>96.68</td>
</tr>
<tr>
<td></td>
<td>2 $\times$ 2</td>
</tr>
<tr>
<td>StressRP (BORP)</td>
<td>91.94</td>
</tr>
<tr>
<td>StrainRP</td>
<td>90.47</td>
</tr>
<tr>
<td>DispRP</td>
<td>37.84</td>
</tr>
<tr>
<td>ALL-EX</td>
<td>26.16</td>
</tr>
<tr>
<td>CST</td>
<td>17.83</td>
</tr>
<tr>
<td>BODT</td>
<td>92.24</td>
</tr>
</tbody>
</table>

- **Maintenance.** A unified implementation automatically weeds out clones.
- **Customability.** Templates can produce optimal and custom elements not obtainable (or hard to obtain) through conventional methods.

A striking example of the latter is the UBOTP macroelement presented in Section A.3 of the Appendix. This concludes a three decade search for a four noded trapezoid which is insensitive to distortion, passes the patch test and retains rank sufficiency. To the writer’s knowledge, this model, as well as its generalization to an arbitrary quadrilateral presented in the sequel cannot be obtained with conventional formulations.

Will the synthesis power translate into teaching changes in finite element courses? This is not presently likely. Two reasons can be cited.

First, advantages may show up only in advanced or seminar-level courses. Beginning calculus students are not taught Lebesgue integration and distribution theory despite their wider scope. Likewise, introductory FEM courses are best organized around a few specific methods. Students must be exposed to a range of formulations and hands-on work before they can appreciate the advantages of unified implementation.

Second, the theory has not progressed to the point where the configuration of a template can be written down from first principles in front of an audience. Only two general rules are presently known: the fundamental decomposition into basic and higher order components, and the procedure to get the matrix structure of the basic component. No general rules to construct the higher order component can be stated aside from orthogonality and definiteness constraints.

How far can templates go? As of this writing templates are only known for a few elements in one and two dimensions, such as beams and flat plates of simple geometry. What is the major technical obstacle to go beyond those? Symbolic power. One must rely on computer-aided symbolic manipulation because geometric, constitutive and fabrication
properties must be carried along as variables. This can lead, and does, to a combinatorial
tarpit as elements become more complicated.

The good news is that computer algebra programs are gradually becoming more pow-
erful, and are now routinely available on laptops and personal computers. Over the next
5 years PCs are expected to migrate to 64-bit multiple-CPUs capable of addressing hun-
dreds of GBs of memory at over 10GHz cycle speeds. As that happens the development
of templates for 3D solid and shell elements in reasonable time will become possible.

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Appendix A. MORE GENERAL PANEL GEOMETRIES

The template framework of four noded membrane elements can be extended to more general geometries, at the cost of increased complexity in symbolic computations. As an aperitif for the sequel this appendix present templates for parallelogram and trapezoidal geometries.

The first-generation (pre-1962) direct elasticity methods recalled in Sections 4–5 do not work properly beyond the parallelogram. The resulting “node collocation” elements fail the patch test and thus cannot fit in the template framework.

Variational methods are required to get stress-assumed and strain-assumed elements that work. For stress elements the Hellinger-Reissner (HR) principle is used. For strain elements, a strain-fit method in conjunction with de Veubeke’s strain-displacement mixed functional is used.

A.1 Parallelogram (Swept) Panel

The geometry of the parallelogram panel shown in Figure 17 is defined by the dimensions $a$, $b$ and the skewangle $\omega$, positive counterclockwise. The template again has the configuration (31) displayed in Figure 6.

With $s = \tan \omega$ the matrices to be adjusted are

$$W = \begin{bmatrix} \frac{1}{a} & 0 \\ -s & 1/b \end{bmatrix},$$

$$H_c = \frac{1}{2ab} \begin{bmatrix} -b & 0 & b & 0 & b & 0 & -b & 0 \\ 0 & -a - bs & 0 & -a + bs & 0 & a + bs & 0 & a - bs \\ -a - bs & -b & -a + bs & b & a + bs & b & a - bs & -b \end{bmatrix}.$$  \hspace{1cm} (63)

The higher order projector $H_h$ is exactly as in (32), whereas $R$ depends on the formulation, as explained below. For future use the compliance and elasticity along the median direction $y_\omega$ (see Figure 17) are denoted by

$$C_{22}^{y_\omega} = C_{22} \cos^4 \omega - 2C_{23} \cos^3 \omega \sin \omega + \left(2C_{12} + C_{33}\right) \cos^2 \omega \sin^2 \omega - 2C_{13} \cos \omega \sin^3 \omega + C_{11} \sin^4 \omega = \frac{C_{22} - 2C_{23}s + \left(2C_{12} + C_{33}\right)s^2 - 2C_{13}s^3 + C_{11}s^4}{(1 + s^2)^2},$$

$$E_{22}^{y_\omega} = E_{22} \cos^4 \omega - 4E_{23} \cos^3 \omega \sin \omega + \left(2E_{12} + 4E_{33}\right) \cos^2 \omega \sin^2 \omega - 4E_{13} \cos \omega \sin^3 \omega + E_{11} \sin^4 \omega = \frac{E_{22} - 4E_{23}s + \left(2E_{12} + 4E_{33}\right)s^2 - 4E_{13}s^3 + E_{11}s^4}{(1 + s^2)^2}. \hspace{1cm} (64)$$

Stress element. A 5-parameter stress element StressPP can be constructed either directly, as done by Gallagher or by the HR principle, starting from the energy-orthogonal stress field

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & y/b & \sin^2 \omega x_\omega \\ 0 & 1 & 0 & \cos^2 \omega x_\omega \\ 0 & 0 & 1 & -\sin \omega \cos \omega x_\omega \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix}, \hspace{1cm} (65)$$
in which \( \sin \omega = \frac{s}{\sqrt{1 + s^2}}, \cos \omega = \frac{1}{\sqrt{1 + s^2}}, \) and \( x_\omega = (x \cos \omega + y \sin \omega)/(a \cos \omega) = (x + ys)/a. \) Both methods give the same stiffness. [Because (65) is an equilibrium field, an equilibrium stress hybrid formulation gives the same answer.] The stiffness is matched by the template with

\[
R_{11} = \frac{1}{3} C_{11}, \quad R_{12} = 0, \quad R_{22} = \frac{1}{3 C_{22}} \frac{1}{(1 + s^2)^2}.
\]

(66)

If the material is isotropic the diagonal entries are \( R_{11} = \frac{1}{3} E \) and \( R_{22} = \frac{1}{3} E/(1 + s^2)^2. \) The Q6 and QM6 elements continue to be clones of StressPP.

**Strain element.** A 5-parameter strain element StrainPP can be constructed by the direct elasticity method of Section 5, or by a variational strain-fitting method\(^{95}\), starting from the companion of (65):

\[
\begin{bmatrix}
e_{xx} \\
e_{yy} \\
2e_{xy}
\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & y/b & \sin^2 \omega x_\omega \\
0 & 1 & 0 & \cos^2 \omega x_\omega \\
0 & 0 & 1 & 0 & -2 \sin \omega \cos \omega x_\omega
\end{bmatrix} \begin{bmatrix}\chi_1 \\
\chi_2 \\
\chi_3 \\
\chi_4 \\
\chi_5
\end{bmatrix}.
\]

(67)

Both methods give the same result. The stiffness is matched by setting

\[
R_{11} = \frac{1}{3} E_{11}, \quad R_{12} = 0, \quad R_{22} = \frac{E_{22}}{3(1 + s^2)^2}.
\]

(68)

**Displacement element.** The conforming, exactly integrated isoparametric element DispPP is matched by setting

\[
R_{11} = \frac{1}{3} \left( E_{11} + 4E_{13}s + (2E_{12} + 4E_{33})s^2 + 4E_{23}s^3 + E_{22}s^4 + (E_{33} + 2E_{23}s + E_{22}s^2)\gamma^2 \right),
\]

\[
R_{12} = \frac{1}{3 \gamma} \left( E_{13} + (E_{12} + 2E_{33})s + 3E_{23}s^2 + E_{22}s^3 + (E_{33} + E_{22}s)\gamma^2 \right),
\]

\[
R_{22} = \frac{1}{3 \gamma^2} (E_{33} + 2E_{23}s + E_{22}(s^2 + \gamma^2)).
\]

(69)

The StressPP element (as well as its clones Q6 and QM6) is again bending optimal along both \( x \) and \( y_\omega \) (median) directions. The symbolic verification is far more involved than for the rectangular element because it requires the use of free-free flexibility methods\(^{96}\) and is omitted.

### A.1 Trapezoidal Panel

The geometry of the trapezoidal panel shown in Figure 18 is defined by the dimensions \( a, b = a/\gamma \) and the two angles \( \omega_1 \) and \( \omega_2 \), both positive counterclockwise. Define

\[
s_1 = \tan \omega_1, \quad s_2 = \tan \omega_2, \quad s = \frac{1}{2}(s_1 + s_2), \quad d = \frac{1}{2}(s_1 - s_2), \quad \phi = bd/a = d/\gamma.
\]

(70)

The template is again given by the matrix form (31). Matrices \( \mathbf{H}_e \) and \( \mathbf{W} \) are as in (63), except that \( s \) has the new definition (70). The higher order projector matrix is

\[
\mathbf{H}_h = \frac{1}{2} \begin{bmatrix} 1 - \phi & 0 & -1 + \phi & 0 & 1 + \phi & 0 & -1 - \phi & 0 \\
0 & 1 - \phi & 0 & -1 + \phi & 0 & 1 + \phi & 0 & -1 - \phi \end{bmatrix}.
\]

(71)
whereas $\mathbf{R}$ depends on the formulation, as detailed next.

**Stress element.** Element StressTP is generated by the 5-parameter stress assumption (65), with one change: the (1,4) entry $y/b$ is replaced by $(y - y_C)/b$. If $y_C = -b^2(s_2 - s_1)/(12a) = -\frac{1}{6}ad/y^2$ the bending stresses are energy orthogonal to constant stress fields. The stiffness matrix derived with the HR principle is matched by

$$R_{11} = \frac{1}{C_{11}(3 - d^2/y^2)}, \quad R_{12} = 0, \quad R_{22} = \frac{1}{3C_{22}^\omega(1 + d^2/y^2)(1 + s^2)^2},$$

(72)

in which $C_{22}^\omega$ is the compliance along the median $y_\omega$ (cf. Figure 18), given by (64).

**QM6 element.** The incompatible-mode element QM6\(^{69}\) is no longer a clone of the stress element unless $d = 0$. Its stiffness is matched by

$$R_{11} = \frac{1}{C_{11}(3 - d^2/y^2)}, \quad R_{12} = 0, \quad R_{22} = \frac{1}{C_{22}^\omega(3 - d^2/y^2)(1 + s^2)^2}.$$

(73)

The only change is in $R_{22}$. The original incompatible-mode element Q6 of\(^{68}\) fails the patch test if $d \neq 0$ and consequently cannot be matched by the template (31).

**Strain element.** Element StrainTP is generated by the 5-parameter strain assumption (67), with one change: the (1,4) entry $y/b$ is replaced by $(y - y_C)/b$. Energy orthogonality is again obtained if $y_C = -b^2(s_2 - s_1)/(12a) = -\frac{1}{6}ad/y^2$. A strain-fitting variational formulation\(^{95}\) yields a stiffness matched by

$$R_{11} = \frac{E_{11}}{3 - d^2/y^2}, \quad R_{12} = 0, \quad R_{22} = \frac{E_{22}^\omega}{3(1 + d^2/y^2)(1 + s^2)^2},$$

(74)

in which $E_{22}^\omega$ is the direct elasticity along the median $y_\omega$ direction, as given by (64).

**Displacement element.** The conforming isoparametric displacement element DispTP with $2 \times 2$ Gauss integration is matched by

$$R_{11} = \frac{E_{11} + 4E_{13}s + s^2(2E_{12} + 4E_{33} + 4E_{23}s + E_{22}s^2) + (E_{33} + 2E_{23}s + E_{22}s^2)y^2}{3 - d^2/y^2},$$

$$R_{12} = \frac{E_{13} + s(E_{12} + 2E_{33} + 3E_{23}s + E_{22}s^2) + (E_{23} + E_{22}s)y^2}{y(3 - d^2/y^2)},$$

$$R_{22} = \frac{E_{33} + 2E_{23}s + E_{22}(s^2 + y^2)}{y^2(3 - d^2/y^2)}$$

(75)
A.2 A Unidirectional-Bending-Optimal Trapezoidal Panel

Element StressTP is $x$-bending optimal as an individual element, but far from it as a repeating macroelement. Consider the configuration of Figure 18(b): two mirror-image trapezoidal elements are glued to form a parallelogram macroelement. The macroelement shape is that of a swept panel, and is obviously repeatable along $x$.

If $a \gg b$ and $s_1 \neq s_2$ the StressTP-fabricated macroelement rapidly becomes overstiff and overflexible in $x$- and $y$-bending, respectively. For example if $a/b = \gamma = 8$, $s_1 = 0$, $s_2 = 1/2$ and isotropic material with $\nu = 1/4$ the bending ratios are $r_x = 11.97$ and $r_y = 0.1414$. For the anisotropic elasticity matrix (61), $r_x = 6.93$ and $r_y = 0.0792$. If an elongated macroelement is supposed to model unidirectional $x$-bending correctly, the overstiffness caused by $s_1 \neq s_2$ is called distortion locking. This phenomenon has been widely studied since the MacNeal-Harder test suite gained popularity.

It is possible to construct a trapezoidal panel that is exact in unidirectional $x$ bending when configured to form a repeatable macroelement as in Figure 18(b), for any aspect ratio $\gamma$ as well as arbitrary side slopes $s_1$ and $s_2$. This template instance will be called UBOTP.

A compact expression is obtained by taking the $R$ matrix of StressTP, generated by (72) and modifying the (2,1) entry of $W$:

$$W = \begin{bmatrix} \frac{1}{a} & 0 \\ C_{11}(3\gamma^2 - ds) + C_{13}(s-d) - C_{12}d & C_{11}(3\gamma^2 - d^2)b \end{bmatrix}$$

It would be equally possible to keep $W$ of (63) and adjust the entries of $R$. However, the correction (76) suggests how this optimality result is extendible to arbitrary quadrilaterals. It is not difficult to prove that $W^T R W$ for UBOTP is positive definite as long as the trapezoid is convex. [Not only that: its condition number is bounded, which is another way of saying that the inf-sup — also called LBB condition — is verified.] Consequently the element stiffness is nonnegative, and has the correct rank.

Figure 19 presents results for a widely used mesh distortion test, which involves one macroelement of the type discussed. Results for six element types: UBOTP, StressTP, StrainTP, DispTP, Q6 and QM6 are shown. The percentage of the correct answer is of course $100/r_x$.

Of these six models only Q6 fails the patch test, but otherwise works better than all others but UBOTP. StressTP, StrainTP and QM6 give similar results, as can be expected, whereas DispTP is way overstiff even for zero distortion. UBOTP gives the correct result for all distortion parameters from 0 through 5, since $r_x \equiv 1$. If the aspect ratio of the
cantilever is changed to, say $2a/b = 10$, the differences between elements become more dramatic.

For distortion performance results on other elements such as Pian-Sumihara and Enhanced Assumed Strain, see Lautersztajn and Samuelsson.72 A penalty-augmented modification of the Pian-Sumihara quadrilateral constructed by Wu and Cheung98 that achieves distortion insensitivity at the cost of rank deficiency (and hence fails the inf-sup condition) is discussed in the sequel.3

At first sight the existence of UBOTP contradicts a theorem by MacNeal,2 which says that four noded quadrilaterals cannot simultaneously pass the patch test and be insensitive to distortion. The escape hatch is that $y$-bending optimality (along the skew angular direction $\omega_1$ of the macroelement) is not attempted. If one tries imposing $r_x = r_y = 1$, the solutions for $\{R_{11}, R_{12}, R_{22}\}$ become complex if $\gamma > > 1$ as soon as $d$ deviates slightly from zero.