A Quick Overview Of Finite Element Templates
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§25.1. Introduction

This Chapter presents a tutorial of the template construction process worked out for a simple 1D element: the 3-node bar. Chapter 28 presents a tutorial based on a 2D element. This gives a better picture of the actual process followed for more complicated elements, it is understandably more difficult to follow, and recommended only for students with medium- to high-level experience in finite element development. The present tutorial is intended for beginners, and focuses on the procedural steps.

The key idea behind templates is divide and conquer. The response of the mechanical system is the sum of several contributions, each of which may be treated by the most convenient method. The method presented here is primarily that of the Free Formulation (FF) of Bergan and Nygård [], the ancestor of templates. Unlike templates, the FF uses assumed displacements throughout. That restriction makes it actually easier to explain.

§25.2. The Example Element

The worked out example concerns the 3-node, 3-DOF, prismatic, linearly elastic bar element of length $L^e$ shown in Figure 25.1(a). The elastic modulus $E$ and cross-section area $A$ are constant over the element. The midside node 3 is placed at the element midspan (center), which is the origin of coordinates. The usual abbreviation for this element in the literature is $\text{Bar3}$. The element node displacement vector is

$$\mathbf{u}^e = [u_1 \quad u_2 \quad u_3]^T. \quad (25.1)$$

The natural coordinate $\xi$ is defined in Figure 25.1(b). It is linked to the Cartesian coordinate by

$$\xi = \frac{2x}{L^e} - 1, \quad \frac{\partial \xi}{\partial x} = \frac{2}{L^e} = J^{-1}, \quad J = \frac{\partial x}{\partial \xi} = \frac{1}{2} L^e. \quad (25.2)$$

The axial displacement field and the axial strain are

$$u = u(x) = u(\xi), \quad e = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = J^{-1} \frac{\partial u}{\partial \xi} = \frac{2}{L^e} \frac{\partial u}{\partial \xi}. \quad (25.3)$$
§25.3. Divide and Conquer: Displacement Decomposition

Decompose the bar motion by splitting the displacement in three parts:

\[ u(\xi) = u_r(\xi) + u_c(\xi) + u_h(\xi). \]  \hspace{1cm} (25.4)

Here \( u_r(\xi) \), \( u_c(\xi) \), and \( u_h(\xi) \) denote the rigid, constant-strain and higher order displacements, respectively. The rigid motion is simply a uniform axial translation. The constant-strain motion corresponds to a uniform stretch of the bar. The higher-order motion is that left upon subtracting the rigid and constant-strain displacements from the actual one.

§25.3.1. Displacement Modes

We express \( u_r(\xi) \), \( u_c(\xi) \), and \( u_h(\xi) \) as scaled polynomials

\[ u_r(\xi) = 1 q_r, \quad u_c(\xi) = \xi q_c, \quad u_h(\xi) = (2 - \xi^2) q_h. \]  \hspace{1cm} (25.5)

The functions 1, \( \xi \) and \( 2 - 3\xi^2 \) are called displacement modes for the rigid, constant-strain, and higher-order axial motions, respectively. The may be thought as generalized, non-interpolatory shape functions. For convenience they are selected so they take the values \( \pm 1 \) at the end nodes 1 and 2. They are pictured in Figure 25.2.

The scalars \( q_r \), \( q_c \) and \( q_h \) are the generalized coordinates. Physically they may be interpreted as modal amplitudes. Later they will be collected into a 3-vector called \( q^e = [q_r \quad q_c \quad q_h]^T \).

Why is the higher order displacement mode taken as \( 2 - 3\xi^2 \)? The answer: orthogonality constraint, is given later.

§25.3.2. Generalized Strains

The axial strains associated to the displacement modes (25.5) are

\[ e_r(\xi) = 0, \quad e_c(\xi) = \frac{\partial \xi}{\partial x} q_c = \frac{2}{L_e} q_c, \quad e_h(\xi) = \frac{\partial (2 - 3\xi^2)}{\partial \xi} \frac{\partial \xi}{\partial x} q_h = -\frac{12}{L_e} \xi q_h. \]  \hspace{1cm} (25.6)

The generalized strain-displacement equations are defined as

\[ e_r(\xi) \overset{\text{def}}{=} B_{rc} q_r, \quad e_c(\xi) \overset{\text{def}}{=} B_{qc} q_c = \frac{2}{L_e} q_c, \quad e_h(\xi) \overset{\text{def}}{=} B_{qh} q_h = -\frac{12}{L_e} \xi q_h, \]  \hspace{1cm} (25.7)

whence

\[ B_{rc} = 0, \quad B_{qc} = \frac{2}{L_e}, \quad B_{qh} = -\frac{12}{L_e} \xi. \]  \hspace{1cm} (25.8)
§25.3.3. Generalized Strains

Since we are dealing with displacement-assumed pieces of the puzzle, the stiffness “matrices” in terms of each generalized coordinate are given by the usual expressions:

\[ K_{rc} = 0, \]
\[ K_{qc} = \int_{-L/2}^{L/2} B_{qc} E A B_{qc} \, dx = \frac{2}{L^e} E A \frac{2}{L^e} L = \frac{4 E A}{L^e}, \]
\[ K_{qh} = \int_{-L/2}^{L/2} B_{qr} E A B_{qr} \, dx = \frac{144 E A}{(L^e)^2} \int_{-1}^{1} \xi^2 J \, d\xi = \frac{144 E A}{(3 L^e)^2} \frac{1}{2} L^e = \frac{48 E A}{L^e}. \] (25.9)

These scalars may be viewed as 1 × 1 matrices for the transformations that follow.

§25.4. Transformation To Physical Coordinates

To combine the effect of the three parts we must transform the generalized stiffness to physical coordinates, namely the nodal displacements \( u_1, u_2 \) and \( u_3 \) collected in \( \mathbf{u}^e \).

§25.4.1. The G Matrix

Evaluate \( u_r(\xi) \), \( u_c(\xi) \) and \( u_h(\xi) \) at the nodes 1, 2 and 3 by setting \( \xi = -1, 1, 0 \), to form the vectors

\[ \mathbf{u}_r^e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_c^e = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_h^e = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}. \] (25.10)

These are pictured in Figure 25.3. Note that these vectors are mutually orthogonal:

\[ \mathbf{u}_r^T \mathbf{u}_c = \mathbf{u}_r^T \mathbf{u}_h = \mathbf{u}_c^T \mathbf{u}_h = 0. \] (25.11)

This is a fundamental constraint for the construction of elements using the template approach, but here it may be just viewed as a recipe. It explains the choice of \( 2 - 3\xi^2 \) as higher order mode made earlier.

Since \( \mathbf{u}^e = \mathbf{u}_r^e + \mathbf{u}_c^e + \mathbf{u}_h^e \), collecting the results of (25.10) we get

\[ \mathbf{u}^e = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} q_r \\ q_c \\ q_h \end{bmatrix} \overset{\text{def}}{=} \mathbf{G} \mathbf{q}^e. \] (25.12)

The 3 × 3 square matrix \( \mathbf{G} \) is assembled by stacking the vectors of (25.10). Its columns are mutually orthogonal, so \( \mathbf{G}^T \mathbf{G} \) will be diagonal. (For more complicated elements, this is the easiest way to verify orthogonality by computer.)
§25.4.2. The H Matrix

The $G$ matrix links $u^e$ to $q^e$. For the transformation to physical coordinates we need its inverse, which is called $H$:

$$H = G^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ -3 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 1/2 & 0 \\ -1/6 & -1/6 & 1/3 \end{bmatrix}. \quad (25.13)$$

The rows of $H$ are mutually orthogonal; thus $HH^T$ is diagonal. The generalized-to-physical transformation is

$$q^e = \begin{bmatrix} q_r \\ q_c \\ q_h \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 1/2 & 0 \\ -1/6 & -1/6 & 1/3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Hu^e = \begin{bmatrix} H_r \\ H_c \\ H_h \end{bmatrix} u^e. \quad (25.14)$$

As shown in the last expression of (25.14), for use in subsequent transformations, it is convenient to row-partition $H$ into three submatrices: $H_r, H_c, H_h$, which are

$$H_r = [1/3 \ 1/3 \ 1/3], \quad H_c = [-1/2 \ 1/2 \ 0], \quad H_h = [-1/6 \ -1/6 \ 1/3]. \quad (25.15)$$

§25.4.3. The Physical Stiffness

We can now use each of the $H$ submatrices listed in (25.15) to transform each generalized stiffness, viewed as a $1 \times 1$ matrix, to physical coordinates:

$$K_r^e = H_r^T K_{qr}^e H_r = 0,$$

$$K_c^e = H_c^T K_{qc}^e H_c = \frac{EA}{L^e} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$K_h^e = H_r^T K_{qh}^e H_h = \frac{4EA}{3L^e} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}. \quad (25.16)$$

We can scale $K_h^e$ by an arbitrary scalar coefficient $\beta$ without altering the orthogonality relations. On the other hand, scaling of $K_c^e$ is not allowed, since if so the constant strain state will not be predicted correctly, impairing convergence.

Adding the contributions we arrive at the final expression for the element stiffness matrix:

$$K^e = K_r^e + K_c^e + K_h^e = \frac{EA}{L^e} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{4\beta EA}{3L^e} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}. \quad (25.17)$$

Remark 25.1. The sum $K_r^e + K_c^e$ is called the basic stiffness matrix in the Free Formulation, and is denoted by $K_b^e$. Evidently $K_b^e = K_c^e$ here because $K_c^e$ vanishes — but see next remark.

Remark 25.2. In linear analysis $K_b^e = 0$ for any element since a rigid motion does not absorb strain energy. But that is not necessarily true in geometrically nonlinear analysis. A body under initial stress can absorb or release energy in a finite rotational rigid motion.
§25.5. Template Nomenclature

The expression (25.17) is called a stiffness matrix template or simply stiffness template. The coefficient $\beta$ is a template parameter. Setting $\beta$ to a numeric value produces template instances. For example,

$\beta = 1$ gives the quadratic isoparametric (isoP) element.

$\beta = 3/4$ gives a macroelement built up with two 2-node bar elements.

$\beta = 0$ gives the 2-node bar (linear) element, which is rank deficient unless node 3 is removed.

The “best” $\beta$ in terms of accuracy can be shown to be $\beta = 1$. Therefore for this element the isoP model is optimal. This is no longer necessarily true for 2D or 3D elements, or for 1D elements in dynamics.

When a template has several parameters, say $\beta_1, \beta_2, \ldots$, a choice of numerical values for them is called a template signature.

Notes and Bibliography

The template approach originally evolved in the late 1980s and early 1990s to construct high-performance stiffness matrices [238,239]. Its roots can be traced back to the Free Formulation of Bergan [83,85,89,90,226], in which the stiffness matrix was decomposed into basic and higher order components. A historical account is provided in a tutorial chapter [253]. The general concept of template as parametrized forms of FEM matrix equations is discussed in [245,251,256].

Mass templates in the form presented here were first described in [246,247] for a Bernoulli-Euler plane beam analyzed with Fourier methods. The study addressed mass-stiffness (MS) pairs. The idea was extended to other elements in [258].

The 3-node bar stiffness template (25.17) was first stated in [238], in which the only free parameter has a slightly different definition. The optimal stiffness matrix for a Timoshenko beam element appeared originally in [798]. A detailed derivation may be found in [630,§5.6]. It is an instance of a template given in [255].

Two powerful customization techniques used regularly for templates are Fourier analysis and modified differential equations (MoDE). Fourier analysis are limited to separable systems but are straightforward to apply, requiring only undergraduate mathematics. (As tutorials for applied Fourier analysis, Hamming’s textbooks [352,351] are highly recommended.) MoDE methods, first published in correct form in 1974 [825] are less restrictive but more demanding on two fronts: mathematical ability and support of a computer algebra system (CAS). Processing power limitations presently restrict MoDE to two-dimensional elements and regular meshes.

The selection of a priori constraint criteria for template free parameters is not yet on firm ground. For example: is conservation of angular momentum useful in mass templates? The answer seem to depend on element type and complexity.