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Axisymmetric Solids, a.k.a. Structures of Revolution
Structures of Revolution are Produced by Rotating a Generating Cross Section Through 360 Degrees

Axis of revolution

Generating cross-section
For Problem to be Axisymmetric, both Loads and Support BCs must be **Rotationally Symmetric**

Illustration for line load $F_r$ and point load $F$
First FEM Analysis of Axisymmetric Solid (Wilson, 1963)

(a) Solid-fuel rocket schematics
(b) Nozzle exit cone
(c) Finite element idealization
"Quasi-Axisymmetric" Marine Structures

(a) (b)
Cylindrical Coordinate System \((r, z, \theta)\)

Global coordinate system  Nonvanishing strains and stresses
A Feature Difference from Plane Stress: the Circumferential or "Hoop" Strain and Associated Stress

The length of the original circumference is $2\pi r$, which grows to $2\pi(r+u_r)$, inducing a hoop strain of $2\pi u_r/(2\pi r) = u_r/r$.
Kinematic Equations (KE)
Strain-Displacement Relations

\[
\begin{align*}
    e_{rr} &= \frac{\partial u_r}{\partial r} \\
    e_{zz} &= \frac{\partial u_z}{\partial z} \\
    e_{\theta\theta} &= \frac{u_r}{r} \\
    \gamma_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = e_{rz} + e_{zr} = 2e_{rz}
\end{align*}
\]

In matrix form
\[
\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \mathbf{D} \mathbf{u}
\]

Note that hoop strain is not given by partial derivative.
**Kinematic Equations (KE)**

**Strain-Displacement Relations**

\[ e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{\theta\theta} = \frac{u_r}{r} \]

\[ \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = e_{rz} + e_{zr} = 2e_{rz} \]

In matrix form

\[
\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \mathbf{D} \mathbf{u}
\]

Note that hoop strain is **not** given by partial derivative.
Constitutive (Stress-Strain) Equations

Ignoring temperature and prestress effects:

$$\sigma = \begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \mathbf{E} \mathbf{e}$$

If material is isotropic

$$\mathbf{E} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 \\ \nu & 1 - \nu & \nu & 0 \\ \nu & \nu & 1 - \nu & 0 \\ 0 & 0 & 0 & \frac{1}{2} (1 - 2\nu) \end{bmatrix}$$

Notice that if $\nu \rightarrow 1/2$ (incompressible material, such as a solid rocket propellant) the foregoing elasticity matrix "blows up"
Equilibrium (Balance) Equations

General (3D) stress equilibrium equations in cylindrical coordinates:

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{r\theta} + \frac{\partial}{\partial z} \sigma_{rz} - \frac{\sigma_{\theta \theta}}{r} + b_r = 0
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{r\theta}) + \frac{1}{r} \sigma_{\theta \theta} + \frac{\partial}{\partial z} \sigma_{zz} + b_z = 0
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{\theta r}) + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{\theta \theta} + \frac{\partial}{\partial z} \sigma_{\theta z} + b = 0
\]

For the axisymmetric problem shear stresses \(\sigma_{r\theta}\) and \(\sigma_{z\theta}\) as well as the hoop body force \(b_\theta\) vanish, and \(\sigma_{\theta \theta}\) is independent of \(\theta\), whence

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rr}) + \frac{\partial}{\partial z} \sigma_{rz} - \frac{\sigma_{\theta \theta}}{r} + b_r = 0
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{r\theta}) + \frac{\partial}{\partial z} \sigma_{zz} + b_z = 0
\]
Total Potential Energy (TPE) Functional in Terms of Original Body Volume and Surface

\[ \Pi[u] = U[u] - W[u] \]

\[ U[u] = \frac{1}{2} \int_V \sigma^T e \, dV = \frac{1}{2} \int_V e^T E e \, dV \]

\[ = \frac{1}{2} \int_V \begin{bmatrix} e_{rr} & e_{zz} & e_{\theta\theta} & 2e_{rz} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} \, dV \]

where strains are derived from displacements (master-field superscript \( u \) omitted to reduce clutter)

\[ W[u] = W_b[u] + W_t[u] \]

\[ W_b[u] = \int_V b^T u \, dV = \int_V \begin{bmatrix} b_r \\ b_z \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} \, dV \]

\[ W_t[u] = \int_{S_t} \hat{t}^T u \, dS = \int_{S_t} \begin{bmatrix} \hat{t}_r \\ \hat{t}_z \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} \, dS \]
Problem Dimensionality Reduction
3D -> 2D

Element of volume:

\[ dV = 2\pi r \, dA \quad \text{reduces} \]

\[ U = \frac{1}{2} 2\pi \int_A r \, e^T E \, e \, dA \]

\[ W_b = 2\pi \int_A r \, b^T u \, dA \]

Element of surface:

\[ dS = 2\pi r \, ds \quad \text{reduces} \]

\[ W_t = 2\pi \int_{s_t} r \, t^T u \, ds \]

Most FEM implementations cancel out the 2\pi factor
Elimination of $2\pi$ Factor Works
Fine Except for Point Load

Line load with components

$$W_F = 2\pi r (F_r u_r + F_z u_z)$$

→ no problem

But watch out for point load $F$ along $z$

Correct work term: $W_F = F u_z$

fits $2\pi$-cancellation if $F$ is divided by $2\pi$: $W_F = 2\pi \left( \frac{F}{2\pi} \right) u_z$