1
Overview
Course Contents

The AFEM course, as configured in its present website, embodies the following parts:

Part 0.  Introduction
Part 1.  Variational Methods in Mechanics (separate web site)
Part 2.  Axisymmetric Solids (transition 2D -> 3D)
Part 3.  General Solids
Part 4.  Advanced Element Derivation Tools
Part 5.  Thin Plates, Membranes, Templates
Part 6.  Shell Structures

Course concludes with student presentations in term projects
Computational Mechanics is an Amalgamation of Four Ingredients
Computer-Based Simulation Stages

Physical system → Mathematical model → Discrete model → Discrete solution

IDEALIZATION

SF → FDM
WF → FEM
VF → FEM

DISCRETIZATION

SOLUTION

Solution error

Verification: discretization + solution error

Validation: modeling + discretization + solution error

RESULT INTERPRETATION
Strong, Weak and Variational Forms of the Mathematical Model

VF

The Inverse Problem

Homogenize variations and integrate

Perform variation(s) and homogenize

SF

Enforce all relations pointwise

WF

Weaken selected relations
Strong, Weak and Variational Forms (Cont'd)

**SF**  **Strong Form.** Presented as a system of *ordinary or partial differential equations* in space and/or time, complemented by appropriate boundary conditions. Occasionally this form may reduce to algebraic equations.

**WF**  **Weak Form.** Presented as a *weighted integral statement* that "relaxes" the point-by-point enforcement of the SF into a domain-averaging statement.

**VF**  **Variational Form.** Presented as a *functional* whose stationary conditions generate the WF and SF.

**Variational Calculus** embodies a set of rules and techniques used to pass from one form to another.
Feasibility of Transformations Between Forms

- VF
  - Usually impossible within SVC
  - Always possible
  - Usually impossible within SVC

- SF
  - Always possible

- WF
  - Always possible
Why Weak and Variational Forms?

The following reasons may be offered:

1. Unification: functional embodies the whole problem
2. Invariance properties wrt changes of coordinate system
3. Provide basis for discrete methods of approximation, notably FEM
4. Characterize overall quantities of interest to engineers
5. Clarify treatment of boundary & interface conditions
6. Permit unified mathematical treatment of questions of existence, stability, error, etc. Also furnish guidelines as to how to achieve desirable "custom" behavior in discrete models
Mathematical Model Forms as Sources of Discretization Methods

- Weighted Residual Methods
- Finite Difference Methods
- Finite Element Methods
- Rayleigh-Ritz Methods

- SF
- VF
- WF

Finite Difference Methods

Weighted Residual Methods

Galerkin
Collocation
Least Squares
Subdomain
Petrov-Galerkin
Discretization Methods

*Universal Methods*
- Finite Difference Methods
- Weighted Residual Methods
  - Galerkin
  - Least Squares
  - Collocation
  - Subdomain

*Special Methods*
- Classical Rayleigh-Ritz
- Boundary Element Methods
- Fluid Volume Methods
- Semi-Analytical Methods

Finite Element Methods derive from classical WRM & RR
The key new ingredient is the choice of approximation spaces
A Simple Example

The problem domain

y(0) = 1

x=0

y(2) = 4

x=2

AFEM Ch 1 – Slide 11
A Simple Example (Cont'd)

**VF**

Functional

\[
J[y] = \int_{0}^{2} \left[ \frac{1}{2}(y')^2 - \frac{1}{2}y^2 + 2y \right] dx
\]

\[y(0) = 1 \quad y(2) = 4\]

**SF**

Boundary value problem

\[y'' = y + 2 \quad \text{in} \ 0 \leq x \leq 2\]

\[y(0) = 1 \quad y(2) = 4\]

**WF**

Variational statement

\[
\int_{0}^{2} r(x) \delta v(x) \, dx + r_0 \delta v_0 + r_2 \delta v_2 = 0
\]

Weighted residual form

\[
\int_{0}^{2} r(x) w(x) \, dx + r_0 w_0 + r_2 w_2 = 0
\]