Participants

Supporting the Transition from Arithmetic to Algebraic Reasoning, STAAR, is a five-year collaborative project of the University of Colorado-Boulder (UCB), the University of Wisconsin-Madison (UWM), and Carnegie Mellon University (CMU).

During year one, the UCB investigative team worked with teachers and students in the Denver Public Schools, the Brighton Community School District, and the Adams 12 School District. These teachers recently adopted one of two reform-based middle school mathematics curricula, and their schools generally serve urban and rural communities with large proportions of low-SES, minority, and ESL families. (For example, Denver is 55% Hispanic, 20% African-American; over 50% of students receive free or reduced-rate lunch.) Various sub-samples of teachers and students participated in interviews, classroom observations, a beliefs survey, and a larger student-data collection effort. The largest student sample ($N = 1313$; with nearly 2000 student assent and parental consent forms administered) came from three middle schools, selected for participation by the Denver Public Schools Office of the Superintendent. Student participation and school (aliases) are broken down as follows:

Total Students = 1313
By School and Grade

<table>
<thead>
<tr>
<th>School</th>
<th>Grade</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>6</td>
<td>101</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>57</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>89</td>
</tr>
<tr>
<td>K</td>
<td>6</td>
<td>193</td>
</tr>
<tr>
<td>K</td>
<td>7</td>
<td>219</td>
</tr>
<tr>
<td>K</td>
<td>8</td>
<td>214</td>
</tr>
<tr>
<td>P</td>
<td>6</td>
<td>145</td>
</tr>
<tr>
<td>P</td>
<td>7</td>
<td>130</td>
</tr>
<tr>
<td>P</td>
<td>8</td>
<td>117</td>
</tr>
<tr>
<td>P</td>
<td>6 and 7</td>
<td>27</td>
</tr>
<tr>
<td>P</td>
<td>7 and 8</td>
<td>21</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>1313</td>
</tr>
</tbody>
</table>

By grade

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>439</td>
</tr>
<tr>
<td>7</td>
<td>406</td>
</tr>
<tr>
<td>8</td>
<td>420</td>
</tr>
<tr>
<td>6 and 7</td>
<td>27</td>
</tr>
<tr>
<td>7 and 8</td>
<td>21</td>
</tr>
</tbody>
</table>
These data are currently being coded for correctness, strategy use, and error type. A subsample of 30 students for each grade for each school (for a total of 270 students) has been selected out for reliability of scoring and coding both at Colorado and Wisconsin.

The UCB team also worked closely with one middle school in the Boulder Valley School District, a higher-SES community, to obtain a range of performance levels among students that supported our early efforts to pilot the assessment items.

In addition, in 2002 we did case studies with 6 teachers in a class of 24 while piloting a teacher professional development (TPD) summer workshop using reform-based curricular materials as a foundational element of the course. We analyzed this pilot study in preparation for the implementation of a similar “Algebra for Teachers” summer course to be taught during year two of the project (July 2003) with 16 teachers currently enrolled.

**Activities and Findings**

*Teaching Experiment*

In two suburban seventh/eighth-grade classrooms (with two control classes) taught by the regular classroom teachers, we conducted an extended (9-week) teaching experiment on beginning algebra and the use of formal representations to describe patterns and solve problems. A summary of the entire 9-week unit is presented in Appendix A. These results are reported in (Nathan, Stephens, Masarik, Alibali, & Koedinger, 2002).

Ninety students in four combined seventh- and eighth-grade mathematics classrooms in Boulder, Colorado, participated in this study for 9 weeks. Two of the classrooms were designated control classrooms and implemented *Connected Mathematics*, the school’s standard curriculum, while the other two classrooms implemented an experimental curriculum called *Bridging Instruction*, which drew on earlier work by some of the research team. For the assessment instrument we used a factorial design to allow systematic examination of the effects of the following elements on student performance: problem linearity (linear or exponential relationship), slope-sign (increasing or decreasing slope), input representation (graphic, symbolic, or verbal expression), and input-to-output translation (graph, symbol, or word-expression input paired with graph, table, symbol, or word-expression output). All problems given in the assessments were first introduced in words. Problem statements then presented the pattern in some “input” representation (i.e., graphic, symbolic, or verbal expression) and asked students to respond to problem-solving items and translation items. The problem-solving items required students to work with a given representation to find a specific value of the dependent variable given a specific value of the independent variable, or vice versa. Translation items required students to represent patterns using a prompted output representation.

Performance in solving problems that provided graphical representations greatly exceeded the performance on all other problem representations. Moreover, students performed better on linear than on nonlinear problems. Students also experienced more success on linear problems when provided a verbal representation than when provided a symbolic one, whereas symbolic representations led to greater success than verbal representations on nonlinear problems. These findings replicate the complexity-representation interaction study of Koedinger, Alibali, and Nathan (1999) showing that verbal representations are most effective when solving problems of lower complexity, whereas symbolic representations are more effective for problems of higher complexity.
Translation among representations appears to be a very advanced skill. At pretest, translation performance was essentially at zero for all but a few types of problems. Students could use preconstructed input representations to solve problems far better than they could generate new representations as part of a translation task. In the most dramatic example, preconstructed graphs were correctly used for problem solving more than 80% of the time, but students across conditions could correctly produce them only about 6% of the time during translation tasks.

Overall, students were more successful on the translation items at posttest than at pretest, though post-instruction performance was still low in both conditions. Students understood and produced instance-based representations (i.e., tables, point-wise graphs) far better than more holistic representations (symbolic and verbal rules, continuous line graphs). Graphs present an interesting case within the instance-based/holistic dimension, because they can be either instance-based (as with scatter plots and bar graphs) or holistic (as with line graphs). This duality does not apply as neatly to symbolic expressions, word expressions, or tables of values. To further understand students’ performance with representations along this dimension, we compared experimental students’ post-intervention abilities to produce accurate graphs when they were judged with instance-based versus holistic scoring criteria. When evaluated from the point-wise perspective, students exhibited relatively high levels of performance. When they were evaluated using holistic criteria, performance was much lower.

Algebra Framework

A review of the literature on algebraic learning and instruction, as well as reflection on preliminary results from our teaching experiment, led us to design a conceptual framework to direct our efforts for developing assessment instruments, for further studying students and teachers in classrooms, and for designing a program and support technology for professional development of middle school mathematics teachers. This framework focuses on a select set of core competencies that mediate how children reason about and use notions of variables and equality. The framework also targets pattern generalization and the use and comprehension of specific formal representations, tables, graphs, words, and symbols; and it enables us to examine how students use these representations to describe patterns and solve problems. Aspects of this framework are summarized in part in Nathan (2002).

This framework has led us most directly to design and implement a set of assessment instruments for use with students in the middle grades in urban and rural areas with a very broad range of mathematical performance across the sample. The design of the representational fluency portion is shown in Appendix B. The final design is shown in Appendix C. An early version of this instrument focusing on representation use was employed to collect the pilot results discussed briefly in the next section.

Measuring Representational Fluency Across Grades

We report on a study of 169 sixth, seventh, and eighth graders in a suburban school district. The results of this study are currently being summarized by Kupermintz and Nathan (in progress). We investigated the effects of presenting patterns with verbal and graphical representations (separately and combined) on student ability to make predictions, create tables, and produce symbolic equations consistent with these patterns. Students received linear patterns in one of three input conditions: verbal, graphical, or combined. Pattern information was presented either as a general rule (holistic) or as a series of values representing instances of the linear pattern (instance-based).
The structure of the assessment follows the forms shown in the first three rows of the design table in Appendix B. Input information about a quantitative pattern was given in graphical or verbal form, or both at once, and the type of information given was either a set of instances (e.g., a set of unconnected points or a verbal list), or a holistic presentation (e.g., a rule or continuous line). Students had to provide near and far predictions that were consistent with the underlying rule, and were asked to describe the rule using mathematical symbols.

A factor analysis revealed that holistic and instance-based items were differentiated as two distinct constructs, suggesting that different cognitive processes may be associated with working in these two modes of problem representation. We found that holistic input resulted in better performance overall. However, instance-based inputs facilitated table generation.

We replicated the verbal-precedence result that influenced much of our earlier work. In a number of tasks, students receiving verbal information outperformed those receiving the equivalent information presented graphically. Generally, the combined (graphical and verbal) representation did not lead to improved student performance. However, we found a trend suggesting that 8th-grade students exhibited better ability in extracting information from combined representations to perform the tasks. Eighth graders who received both verbal and graphical information performed better than 8th graders who received either type of input separately. In contrast, sixth and seventh graders showed a trend suggesting that they performed better when they received either verbal or graphical input alone, compared to the combined representation. This pattern was strongest when the information was presented in a holistic manner. We discuss our findings in terms of the affordances provided by different input representations relative to the particular tasks students tackled, and the implications of these affordances for the construct of representational fluency in particular and for development over the middle school years of algebraic reasoning in general.

Item-Level Meta Analyses

In order to enhance our ability to generalize across separate assessment administrations, both within this grant and across studies, we have developed an item-level database. Still in its prototype stage, this database will allow us to perform item-level “meta analyses” across data sets. These analyses would inform our cognitive and developmental models of the ways in which the features of a wide range of algebraic activities influence performance, errors, and strategy use. We describe the system design and functionality briefly.

The Study Data Analyzer is a tool for exploring aggregate student performance on test questions. Currently, the tool is a Web application that allows users to customize predefined views of the study data. As data from many studies are added into the tool’s flexible architecture, the power of the dataset grows, and the analyses extracted from it will have correspondingly more informative results.

This tool was created to provide data analysis of a larger scope than possible from any single study’s results. To populate the database of student data, student tests are scored for success and strategies used, then aggregated by grade. The aggregate data are entered into the database and become immediately available for subsequent searches. The Study Data Analyzer includes five prestructured views into the data, three of which may be customized to enable examination of specific problem types, strategies, and test questions. Users are first presented with a Search page. The first option on the Search page is “Show me the likelihood by grade of correctly solving a problem of type [Types].” The content of the [Types] pull-down is a dynamically generated list of all unique problem types in the current database. Results are ranked by student grade level and include a percentage-based measurement of score (percentage correct for all matching problems) and the power of the matched problems in the database. The second search
option is “Show me the likelihood by strategy of correctly solving a problem of type \( [\text{Types}] \).” The third search option is “Success rates for all problem types” and yields a list, ranked by percentage-based score, of all problem types encountered in the database. The fourth search option is “Success rates for all strategies” and yields a list, ranked by percentage-based score, of all strategy codes encountered in the database. If a researcher knows of a specific item and wishes to examine the strategies used to solve it, the final search option, “Show me the strategies used for the question named \( [\text{Problem Label}] \), ranked by \( [\text{Grade/Code/Score}] \),” provides those data.

The Study Data Analyzer tool consists of several pages of PHP code served to the Web via an Apache Web server. The data are stored in and accessed through a MySQL database. The tool is available to researchers—with password protection—via any standard Web browser. PHP, Apache and MySQL are open-source applications, which together provide an industry-standard, efficient programming environment for database-driven Web applications.

Classroom Observations

Guided by the framework for algebra, a classroom observation protocol was developed that focuses on the following questions:

- Equality: Is use primarily for operations or relational?
- Variables: Are variables used primarily as placeholders ("unknowns") or as varying values?
- Patterns and Functions: Is the pattern or function primarily unidimensional or of two or greater dimensions?
- Formal Representations: What formal representations are in use: symbolic, graphical, verbal, tables of values?
- Representation Translation: At what stages does the problem solving move among different formal representations?
- Problem Solving: When problem solving occurs, is the primary aim to achieve an answer to the posed problem, or to discuss and reflect upon or compare strategies?
- Procedural and Conceptual Knowledge: Does the introduction of new topics focus on procedural aspects or conceptual aspects, or is there a balance between the two?
- Algebraic Corridor: Are clear references made about how a given activity or concept will connect to later algebra instruction? (Such references are especially useful for sixth-grade classes that have not yet had any formal algebra.)

We used this framework to observe seven class sessions (in the classrooms of two DPS teachers) in year one, and we plan to use it in the mathematics classrooms of six teachers in year two. Preliminary results indicate that making connections to algebraic concepts, or generalizing core concepts, occurs rarely in non-algebra units.

Teacher Practices and Beliefs

As planned for year one, we conducted two studies to gain a better understanding of contemporary issues regarding the teaching practices, belief structures, and content knowledge of middle school teachers engaged in algebra instruction. In the first study, the UCB research group explored the notion of teachers' tolerance for discomfort (TTD) as it pertains to the teaching and learning of algebra. Building on an existing study of TTD (Frykholm, in press) we collected data with the intent of illuminating the discomfort teachers experienced as they attempted to align their teaching practices with those endorsed by the (newly adopted) reform-oriented algebra curricula they were implementing. In accordance with the framework we adopted, we gave specific focus to issues associated with the teaching of patterns and
functions. Six teachers were each observed for a minimum of six classroom lessons during the academic year. Additionally, each of these teachers participated in at least two formal interviews, belief surveys, pre- and post-lesson conferences, and numerous informal conversations. Findings of this study illustrate connections between teacher uncertainty and content domains, and they reveal the impact of that uncertainty on teaching practices and beliefs. These findings are elaborated in a paper recently accepted for publication in *The Journal of Curriculum and Supervision* (Frykholm, in press).

In our second study under the umbrella of teacher practices and beliefs, we conducted a pilot summer workshop devoted to professional development of middle school teachers. In particular, the impact and potential of content-based professional development were under investigation as we documented and analyzed experiences of teachers who attended our summer school courses for credit. Of the twenty-four students enrolled in the class, eight were from the Denver/Brighton area. As highlighted below, detailed case studies for six of these students were developed.

One set of analyses explored alignments and/or contrasts between the stated beliefs of the participants and the premises underlying the orientation and structure of both the two mathematics curricula used as the basis for the course and the curriculum programs implemented in the teachers’ own classrooms. For example, several teachers in the study expressed beliefs that students should first learn symbolic manipulation of standard algebra problems before attempting conceptually based problem contexts. This belief stands in contrast to the perspectives of one of the curriculum programs used heavily in the content courses, *Mathematics in Context*, a program that supports the development of conceptual understanding of algebra on the basis of verbal rather than symbolic precedence. The observations and interviews of these teachers suggest they experience difficulty and uncertainty as they try to reconcile their beliefs with the premises of the curricula and the actions of their students. The reflections of these teachers—not only about their teaching but also about their “re-learning” of algebra—have been particularly insightful. We also developed two case studies in more detail, in order to examine more closely the impact of the courses on teachers’ beliefs and content-knowledge structures. This analysis resulted in the development of a framework for professional development centered upon the notion of teachers’ “turning points” in both belief and practice. These findings are summarized in an article currently under review by the *Journal for Research in Mathematics Education* (Frykholm & Pittman, under review). In this article we define and illuminate turning points, and we discuss their promise both for stimulating change in teacher practice and for establishing guidelines for the professional development of middle school teachers.

**Preliminary Work on Teacher Professional Development**

Data from this pilot workshop and from the assessments of students’ algebraic reasoning are presently being considered by the project’s teaching and professional development tiers as they jointly prepare for the educational and research components of the Algebra for Teachers program to be conducted at UCB during the summer of 2003.

We have drafted a conceptual framework to guide and inform the design and evaluation of large-scale professional development. The framework draws from cognitive, social constructivist, and sociocultural learning theories; empirical research on teacher learning (Putnam & Borko, 1997, 2000; Greeno, 1998; Krajcik et al., 1994); teachers’ perspectives on their own professional development needs (Garet et al., 2001); and recent scholarly analyses of systemic issues in implementing large-scale change (Blumenfeld et al., 2001). STAAR researchers at UCB are currently using this framework to guide the design of TPD interventions. We also intend to use the framework to guide future studies of the impact of STAAR TPD on teachers’ learning and practice.
The central features of the framework focus on

- distinction between substance (content) of teacher learning and process of teacher learning;
- influence of prior knowledge and beliefs on learning;
- knowledge as active, socially constructed, and distributed across communities of learners and cognitive artifacts;
- knowledge as situated in the context in which it is constructed;
- importance of a coherent approach to TPD in which the goals of TPD are explicitly aligned with district, state, and national standards and with teachers' personal professional-development goals;
- value of case-based approaches to TPD;
- role of systemic influences (such as school-based and district-based administrative and technological support) in large-scale TPD initiatives;
- influence of incentive to learn on teacher participation, learning, and change;
- methodologies for assessing teacher learning and change.

We expect that online learning tools will be key aspects of large-scale TPD efforts, and we are in the process of working in conjunction with our colleagues at the University of Wisconsin to design online tools for use in the STAAR TPD interventions. Our conceptual framework has enabled us to review and analyze existing online TPD sites in terms of the extent to which various features built into these learning environments might effectively support teacher learning and change. Our analysis is similar to analyses recently produced by Barab et al. (2002) and Riel and Polin (2001). Each of these analyses focuses on one TPD site and draws primarily on “communities of practice” models for teacher learning. Our work looks across a wider range of online environments and brings a broader range of perspectives on learning into the analysis. Additionally, our analysis has evolved into a recursive process of evaluating the sites and revising our framework on the basis of insights about the value of design features observed on these sites.

To date, we have looked most closely at commercial, widely accessible products available to in-service teachers, including Teachscape, LessonLab, and TappedIn. Our findings suggest that online TPD tools and processes are currently well developed in some areas, and in need of development in other areas. Substantial progress has been made toward

- anchoring teacher learning activities in the use or development of cognitive artifacts (e.g., curriculum, student work, videotapes of teaching);
- situating learning in context (e.g., via the use of multimedia cases and learning activities linked to teachers’ actual classroom practices);
- strengthening the alignment of teachers’ learning activities with district, state, and national teaching standards; and
- developing the knowledge base for teaching and providing access to collective bodies of knowledge about teaching (e.g., print resources, multimedia cases, teacher discussion forums).

Key areas in need of development include

- expanding the availability of sound conceptual frameworks for large-scale TPD that can be used to guide the design and implementation of online tools;
- finding effective approaches to fostering socially interactive online learning;
- promoting multimedia case design that is explicitly guided by well-developed models for teacher learning;
- exploring techniques for assessing teacher learning and change;
- cultivating methods of providing incentive for teachers’ full engagement in learning and change processes;
developing systems for providing cultural, administrative, and technological support for teacher participation in TPD programs; and

establishing a knowledge base for TPD to support and inform the practice of professional development providers.

Appendix A: Outline of 9-Week Bridging Instruction Algebra Unit Plan for 7th/8th Grade

Pre-Intervention Assessment (2 days)

Activity 1 Bridges and Pennies

Lesson 1
Set up experiment; collect data; organize data and share with group members. Make different bridges with equal length. Stack pennies and fill grid cells (graph paper).

Experiment:
Different groups (a) cup with thin paper; (b) cup with thick paper; (c) no cup with thin paper. Try 6 groups, 2 out of the 3 conditions. Conduct 4 trials using maximum of 18 sheets of paper, will roll die to determine thickness of bridge, no repeats within a group.

Homework: Organize data.

Lesson 2
Groups discuss data organization; in terms of organization combine across partner groups (those that did the same experiment); tweak tables. Whole class discussion on organization (with overheads or big chart paper).

Homework: Write a verbal rule describing relationship between the condition of the bridge and the number of pennies, draw a pictorial representation of the relationship, and make a formula that describes the relationship drawn.

Lesson 3
In groups: Create group.
- Construct graph (picture), verbal representation, and formula (from homework).
- Meet with like-condition groups and tweak representations.
- Public display.
- Whole class discourse on ‘best’ & ‘good’ representations (conventions).
- What makes good representations? Title graphs, name variables, etc.: ‘pennies per sheet.’

Homework: Make predictions for different values (missing values – thickness of bridge? big and fractional thicknesses), be able to explain and defend predictions.

Lesson 4
Teacher-led discussion on changing conditions and impact on changing representations. Predict for (a) no cup/thick paper (b) prove it – through mathematics and experimentation. Accuracy of representations – do they describe the data? Intersection – how do you get the ‘thin’ line to hit the ‘thick’ line?

Homework: \[ y = b - mx \] problem and \[ y = mx + b \]. Triangle and Pool Problem (possibilities).
Given data tables, make graph, verbal relationship, and write a formula.
Lesson 5  Abstraction and closure: Review homework.  This is a linear function.  Lines have 2 and only 2 components (parameters – slope and intercept).  What if's? Different cup, different paper, what if given that the bridge will hold 43 pennies, how thick? Does it have a cup?

Letter writing to have students reflect on what they did, how they interpret it and what they learned.

Activity 2  Bridges and Pennies / Law of Torque

Lesson 1  Repeat Bridge and Pennies Problem but use varying lengths of the bridge, 3 inches to 11 inches. See as a series of lines with changing slopes, look at equations of 3 lines, therefore rate of change of variables.

Lesson 2  Complete work from Day 1.

Lesson 3  Discuss Law of Torque (nonlinear relationship).  Provide balance scale (one per group?).  Collect and organize data – find sets of weight and locations that balance.  For example: If given a weight of 20 and position 6, what would balance?  120 at 1, 60 at 2, 50 at 2.4, 40 at 3, 30 at 4, 20 at 6, 10 at 12, 5 at 24, 1 at 20.  Construct table, analogue, graph.  Each group would be given one of the pairs and asked to find the others, that way all graphing the same set of data.

Discuss choices of independent – dependent variables (arbitrary but consistent).  Graphs pairs that balance.

Homework:  Finish group work, propose a verbal rule and a formula.

Lesson 4  Representation of rules.  Relationship of rule to graph – pairs represent equal areas.  Curve of all, balanced state (that is, curve of all equal areas) and qualitative interpretation.  Give groups different fixed torques to match.

Homework:  Letter writing to summarize work.

Activity 3  Building Squares

(Developing graphing conventions with use of analogue)

Lesson 1  Build squares with given length (result of tossing pair of dice).  Make an analogue and table (will use graph paper that is 12 x 120 inches so students can see a one to one correspondence).

Question:  Consider area in terms of number of tiles and square measure; what if tile is not of unit length?
Lesson 2  
Transfer analogue and table to a graph on regular graph paper. 
Graphing vocabulary: axes, independent and dependent variables, ordering of number 
pairs.  
How is analogue related to graph?

Lesson 3  
Discuss verbal and symbolic rules.  
Fluency between representations

Lesson 4  
Vary the problem – work with rectangles. $x(x + 2)$ and $(x – 1)(x + 1)$.  
Build, analogue, table and graph?

Lesson 5  
Midterm Assessment

Activity 4  
**Footrace with Handicap (distance = rate x time)**  
*(CMP – Moving Straight, pp. 21–23)*

Objectives: Construction and interpretation of graphs and symbolic representation.  
Interpretation of slope (rate of change) and intercept.  
Treat each variable as independent.  
- If only have 1 minute for race, how far a lead?  
- If only x distance, how much lead?  
- Play with lead-start (time/distance) and see where they meet.

Lesson 1  
Present the word problem.  
Solve in any manner, explain solution and why it works.  
Predict, how do students assess how good is their prediction?  
What makes a close race?  
Understand nature of problem, for constant rate of change.  
Graph function that start at position and 50-meter head start.  
Where do they begin?  
Why different slope?  
What does it mean when the lines intersect?  

Homework: Construct verbal rule and equation that represents each person in the 
race. Why is the graph evidence?

Lesson 2  
Discuss verbal and symbolic representation from homework.  
Discuss elements of the equation – rate of change and starting position.  
Discuss how to utilize general graph.  
Identify more than one way to generate graph, depending on what is given.  

Homework: 4 to 6 problems. Graphs of different equations, need to identify positions 
and walking rate of the participants. Include another situation (cost and donuts) where 
students need to identify parts of graph (2 graphs with axes interchanged).

Lesson 3  
Generate table, equation, and graph based on modifying conditions.  
Starting position changes or time that participants start.  
Graph can predict length of race or time of race (use vertical and horizontal lines about 
the point of intersection).
Lesson 4  Collect own data, determine rate of walking.  
Pick partner, who wins, work up 2 different sets of race condition (equation, verbal, table, graph).

Homework: Finish write-up of finding.

Lesson 5  Presentation of Partner Work

Activity 5  Exponential Decay and Growth

Lesson 1  Paper cutting exercise to demonstrate exponential decay of area.

Use a 9 x 9 piece of construction paper, measure it.  
Cut piece in half, new piece in half, etc.  
Label each piece with the area (measure or do mathematically).  
Analogue – map values onto graph paper by shading.  
- Stripe the area of each piece and stack.  
- Or adjust the top of each piece to a vertical scale that represents area.  
Have the conversation that emphasizes that we are measuring area.  
Analogue is a precursor to the graph.  
Make a table.

Homework: What happens to the area of the piece after the 12th cut, the 15th cut, the 100th cut? How many cuts until the area of the piece is 0 square units?

Important concepts: Area approaches zero, always 1/2 the previous area, never becomes negative, we can keep asking area theoretically even if piece does not exist in real life.

Lesson 2  Another 9 x 9 piece of construction paper, decrease area by 2 square inches.  
Prepare analogue – shade in square on graph paper.  
Make table, graph.

Graph the information from Day 1 on the same axis as the y = b – mx.  
Discussion of (linear) absolute amount of decrease (every event take away 2 square units) versus (exponential) relative amount of decrease (for every event take away 1/2 of the previous value).

Write equation for linear relationship.  
Looking at the similarity between 2x, x^2, 2^x, (1/2)x:  
Construct the symbolic equation for the exponential function.

Homework: Summarize similarities and differences between the two situations (graphs, tables, events, symbolic).

Lesson 3  Summarize differences from Day 2 – symbolic → table and use of calculator. Use exponential growth diffusion to spread the TI 82 program that will be used to illustrate the exponential decay and produce a table.
Lesson 4  Exponential growth. Start with discussion of exponential diffusion that took place in class on Day 3.

Use of Rice Krispies bars to experiment with exponential growth by cutting the bar in half then stacking, cut in half, stack, etc.

Analogue – shade squares of graph paper.
Table, make predictions because of the restriction of the Rice Krispies bars.
Graph. TI 82 Graphing Calculator.
Write symbolic representation. Symbolic comes from verbal or computer program.

Lesson 5  “Power of Ten” video.
Appreciation of exponential growth and decay.
Letter – similarities, differences, similar to discussion on Day 3.

Activity 6  Cube Problem

   Question: What is the needed information to solve the problem? Manipulatives, definition of terms (interior, corner, edge, faces, total number of cubes).

Lesson 1  Present the problem: Students build cubes of side length 2 – 5, recording number of corners, edges, faces, and interior cubes.

   Homework: Predict values of each for the next 2 cases; opportunity to think about rules for each (interior, face, edge, corner).

Lesson 2  Take students’ work → symbolic representations for each.

Lesson 3  (TI 82 Graphing Calculator)
   Verbal → graphing – look at varying rates of change for each part.
   → sum of y values (use graphs) equals total number of cubes.

Lesson 4  Predict number of type of cube when length of side = 10.
Look at symbolic representations, use values, add them to get total number of cubes.
Use symbolic substitution to combine like terms = n³

Lesson 5  Look at the case when 0 cubes, and when 1 cube.
Examine graph, table, symbolic-representation-use substitution.

Final Assessment (2 days)
### Appendix B: Design of Representational Fluency Portion of Assessment Instrument (Questions 7 & 8)

<table>
<thead>
<tr>
<th>Form (Repeated Measure)</th>
<th>Input Representation</th>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Output Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A √</td>
<td>Graph</td>
<td>Instance</td>
<td>Holistic</td>
<td>Symbol</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CD 3X + 2</td>
<td>Video 2X + 3</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Verbal</td>
<td>Holistic</td>
<td>Instance</td>
<td>Symbol</td>
</tr>
<tr>
<td></td>
<td>Shuttle 2X + 3</td>
<td>Yard Work</td>
<td>CD 3X + 2</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>G + V</td>
<td>Holistic</td>
<td>Instance</td>
<td>Symbol</td>
</tr>
<tr>
<td></td>
<td>Yard Work 3X + 2</td>
<td>Shuttle</td>
<td>CD 2X + 3</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>Symbol</td>
<td>Holistic</td>
<td>Instance</td>
<td>Graph</td>
</tr>
<tr>
<td></td>
<td>Yard 3X + 2</td>
<td>Shuttle</td>
<td>CD 2X + 3</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>Verbal</td>
<td>Instance</td>
<td>Holistic</td>
<td>Graph</td>
</tr>
<tr>
<td></td>
<td>CD 3X + 2</td>
<td>Video 2X + 3</td>
<td>CD 3X + 2</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>V + S</td>
<td>Instance</td>
<td>Holistic</td>
<td>Graph</td>
</tr>
<tr>
<td></td>
<td>Video 2X + 3</td>
<td>CD 3X + 2</td>
<td>CD 2X + 3</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>Graph</td>
<td>Holistic</td>
<td>Instance</td>
<td>Verbal</td>
</tr>
<tr>
<td></td>
<td>Shuttle 2X + 3</td>
<td>Yard 3X + 2</td>
<td>CD 3X + 2</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>Symbol</td>
<td>Instance</td>
<td>Holistic</td>
<td>Verbal</td>
</tr>
<tr>
<td></td>
<td>Video 2X + 3</td>
<td>CD 3X + 2</td>
<td>CD 2X + 3</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>S + G</td>
<td>Instance</td>
<td>Holistic</td>
<td>Verbal</td>
</tr>
<tr>
<td></td>
<td>CD 3X + 2</td>
<td>Video 2X + 3</td>
<td>CD 3X + 2</td>
<td></td>
</tr>
<tr>
<td>Count</td>
<td></td>
<td>I CD - 3</td>
<td>I CD - 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>I Vid - 2</td>
<td>I Vid - 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>I NC1 - 0</td>
<td>I NC1 - 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>I NC2 - 0</td>
<td>I NC2 - 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>H CD - 0</td>
<td>H CD - 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>H Vid - 0</td>
<td>H Vid - 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>H NC1 - 2</td>
<td>H NC1 - 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>H NC2 - 2</td>
<td>H NC2 - 0</td>
<td></td>
</tr>
</tbody>
</table>
Appendix C: Design of Entire Assessment Instrument Used for Longitudinal Study

1. Parts A, B, C: Meaning of equality symbol (“=”).
2. Uses and meaning of variable in number sentences.
5. Justification and explanation of statements about equations with symbolic representations of variables.
6. Justification and explanation of statements about equations with variables in a situation.
7. Representational fluency question 1 (see Appendix B).
8. Representational fluency question 2 (see Appendix B).
9. Problem-solving question 1 (from Carnegie Mellon)
11. Problem-solving question 3 (from Carnegie Mellon)

Opportunities for Training and Development

Graduate students participating in this research project gain experience for several aspects of contemporary educational research. Students gain experience on the design of large-scale assessment instruments and their administration, which includes working with school and district administrators as well as teachers. Data handling of a longitudinal data set to ensure confidentiality, data coding and data entry are also part of the experience for students. Data analysis from both quantitative (statistical analyses for accuracy and developmental trends) and qualitative (strategy and error analyses) perspectives are also a central part of this project, as is the integration of these perspectives to tell a rich story of student reasoning and development.

Graduate students working in the classrooms gain experience and skills in conducting classroom observations and designing and administering interview protocols to students and teachers. Graduate students also learn how to analyze data from classroom discourse and interview responses, and to interpret these data in light of our theoretical framework and other data analysis activities.

Graduate students working with teachers participate in the development and implementation of both face-to-face and on-line teacher professional development programs. This is especially valuable for those graduate students on the project who have been teachers and are now working to enhance the professional development of others. Much of this work also stems from the Design Experiments tradition. Skill development in this area in very attractive for new graduates and will make them highly suitable for future research and teaching positions.

Outreach

Springing from the large number of requests for TPD support by teachers and schools in the area, a collaboration between Tier 2 and Tier 3 has emerged over the past several months. The intent of this collaboration is to continue to heighten the presence and impact of the STAAR project in schools in our surrounding community. Our first formal effort in this regard is to design and implement TPD opportunities for practicing teachers. The success of the pilot program in the summer of 2002 has prompted our collective work toward a new program for 2003.
Publications and Products

Papers


Presentations

Nathan, M. J. (2002, December). Discussant for “What Do We Know and Need to Know About Facilitated Online Learning for Teacher Professional Development.” (Sharon Derry, Chair). Wisconsin Center for Education Research at the University of Wisconsin-Madison.


Books or Other Nonperiodical, One-Time Publications


What Web Site or Other Internet Site Have You Created?

[http://algebra.colorado.edu](http://algebra.colorado.edu)
Other Specific Products

The above discussion of the Study Data Analyzer describes a tool we are developing for exploring aggregate student performance on test questions. Currently at a proof-of-concept stage, the tool is a Web application that allows users to customize predefined views of the study data. As data from many studies are added into the tool’s flexible architecture the power of the dataset grows, and the analyses extracted from it are expected to have correspondingly compelling results. The Study Data Analyzer offers a new vision of data analysis across data sets (meta-analyses) at the item level, and across studies by matching items according to relevant underlying design features and comparing performance across student demographics.

Contributions

Our work stands to make a contribution to the work of mathematics teacher educators who find themselves focused on issues of teaching as they pertain specifically to the middle school algebra classroom. This work is intended to help map the terrain of middle school mathematics teaching and learning, with a particular focus on the role that innovative curricular materials play (through both their positive contributions and the challenges they present) in the present era of educational reform. Ultimately, we hope this work will provide clarity and insight for the design of TPD opportunities for practicing teachers.

References