Problem #1 (10 points): Express each of the following complex numbers in polar exponential form:

(a) \( \frac{1}{\sqrt{2}} - i/\sqrt{2} \)
(b) \( -\sqrt{3} + i \)

Solution: Let \( n \) be an integer.

(a) \( r = 1 \) and \( \arg = \frac{7\pi}{4} \), so \( \frac{1}{\sqrt{2}} - i/\sqrt{2} = e^{i\frac{7\pi}{4} + 2n\pi} \)
(b) \( r = 2 \) and \( \arg = \frac{5\pi}{6} \), so \( -\sqrt{3} + i = 2e^{i\frac{5\pi}{6} + 2n\pi} \)

Problem #2 (10 points): Solve for all the roots of the following equation: \( z^4 + 2z = 0 \).

Solution: \( z(z^3 + 2) = 0 \), so \( z = 0 \) or \( z^3 = -2 \), and then \( r^3 = 2 \), \( e^{3i\theta} = e^{i\pi} \implies \theta = \pi/3 + 2\pi n/3 \), \( n = 0, 1, 2 \).
Thus, the roots are \( z = 0, 2^{1/3}e^{i\pi/3}, 2^{1/3}e^{i\pi} = -2^{1/3}, 2^{1/3}e^{5i\pi/3} \).

Problem #3 (10 points): Express each of the following in the form \( a + bi \), where \( a \) and \( b \) are real:

(a) \( \frac{1}{1+i} \)
(b) \( (1+i)^3 \).

Solution:

(a) \[ \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1}{2} - \frac{i}{2} \]
(b) \( (1+i)^3 = 1+3i+3i^2+i^3 = -2+2i \)

Problem #4 (12 points): Establish the following results:

(a) \[ |z - w| \leq |z| + |w| \]
(b) \[ |w\bar{z} + \bar{w}z| \leq 2|wz| \]

Solution:

(a) \[ |z - w|^2 = (z-w)(\bar{z}-\bar{w}) = |z|^2 + |w|^2 - (w\bar{z} + z\bar{w}) = |z|^2 + |w|^2 - 2\text{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2. \]

Problem #5 (18 points): Sketch the region associated with the following inequality and determine if the region is open, closed, bounded or compact:
\[ 1 \leq |z + 4| \leq 3; \text{ is this region connected? Explain.} \]

Solution: \( 1 \leq |z + 4| \leq 3 \iff \frac{1}{3} \leq |z + 2| \leq \frac{3}{2} \) is an annulus centered at \( z = -2 \) with inner radius \( \frac{1}{3} \) and outer radius \( \frac{3}{2} \). This region is closed, bounded, therefore compact; it is connected.

Problem #6 (10 points): Show that
\[ \frac{1}{z} = \frac{\bar{z}}{|z|^2}. \]
Use this formula to show that \( \text{Re}(1/z) \) and \( \text{Re}z \) have the same sign for all \( z \).

Solution:

\[ \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2}. \]
Solution:

\( \operatorname{Re} \left( \frac{1}{z} \right) = \frac{\overline{z}}{|z|^2} = \frac{x}{|z|^2} \)

and its sign is equal to \( z = \operatorname{sign} \operatorname{Re} z \).

**Problem #7** (12 points): Use the series representation

\[ e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad |z| < \infty \]

to determine the series representations for the following function: \( \cosh z \). Use this result to deduce where the power series for \( \operatorname{sech} z \) would converge.

**Solution:** Since \( 1 + (-1)^j = 0 \) when \( j \) is odd, we get that

\[
\cosh z = \frac{e^z + e^{-z}}{2} = \frac{1}{2} \left( \sum_{j=0}^{\infty} \frac{z^j}{j!} + \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!} \right) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.
\]

Since the series for \( \cosh z \) converges in the whole finite \( z \)-plane, the series for \( \operatorname{sech} z = 1/\cosh z \) converges in a circle around \( z = 0 \) with the radius being distance to the closest point(s) where \( \cosh z = 0 \), i.e. \( z = \pm i \pi/2 \), i.e. circle \( |z| < \pi/2 \).

**Problem #8** (10 points): Consider the following transformation

\[ w = \frac{az + b}{cz + d}, \quad \Delta = ad - bc \neq 0. \]

Show that the map can be inverted to find a unique (single-valued) \( z \) as a function of \( w \) everywhere.

**Solution:**

\[ w(cz + d) = (az + b) \implies z = \frac{b - d w}{cw - a}, \]

valid everywhere if one considers extended \( z \)-plane. Then \( z = -d/c \) maps to \( w = \infty \) by the direct map, and \( w = a/c \) maps to \( z = \infty \) by the inverse map.

**Problem #9** (12 points): To what curves on the sphere (in Fig. 1.2.6) do the lines \( \operatorname{Re} z = x = 0 \) and \( \operatorname{Im} z = y = 0 \) correspond?

<table>
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<tr>
<th><strong>Problem #10</strong> (20 points): Evaluate the following limits:</th>
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<tr>
<td>(a) ( \lim_{z \to 0} \frac{\sin 3z}{z} )</td>
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<tr>
<td>(b) ( \lim_{z \to 0} \frac{\sin 3z}{\sin z} )</td>
</tr>
<tr>
<td>(c) ( \lim_{z \to \infty} \frac{z^2 + z}{(2z + 3)^2} )</td>
</tr>
<tr>
<td>(d) ( \lim_{z \to \infty} \frac{\sin 2z}{\cosh^2 z} )</td>
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</table>

**Solution:**

(a) \( \lim_{z \to 0} \frac{\sin 3z}{z} = 3 \lim_{z \to 0} \frac{\sin 3z}{3z} = 3. \)

(b) \( \lim_{z \to 0} \frac{\sin 3z}{\sin z} = 3 \lim_{z \to 0} \frac{\sin 3z}{3z \sin z} = 3. \)

(c) \( \lim_{z \to \infty} \frac{z^2 + z}{(2z + 3)^2} = \lim_{z \to \infty} \frac{z^2(1 + 1/z)}{4z^2 + 12z^2 + 9z^2} = \frac{1}{4}. \)

(d) \( \lim_{z \to \infty} \frac{\sin 2z}{\cosh^2 z} = \lim_{z \to \infty} \frac{2 \sinh z}{\cosh z} = \lim_{z \to \infty} \frac{2(e^{z+iy} - e^{-z-iy})}{e^{z+iy} + e^{-z-iy}}, \)

so if \( e \) \( x \to +\infty \), one gets 2, while if \( x \to -\infty \), one gets \(-2 \), so the limit does not exist.

**Problem #11** (12 points): If \( |g(z)| \leq M, M > 0 \) for all \( z \) in a neighborhood of \( z = z_0 \), show that if

\[ \lim_{z \to z_0} f(z) = 0, \]

then

\[ \lim_{z \to z_0} f(z) g(z) = 0. \]

**Solution:**

\[ |f(z) g(z)| = |f(z)||g(z)| \leq M |f(z)| \implies \lim_{z \to z_0} |f(z) g(z)| \leq M \lim_{z \to z_0} |f(z)| = 0, \]
therefore \( \lim_{z \to z_0} |f(z)g(z)| = 0 \), which means also \( \lim_{z \to z_0} f(z)g(z) = 0 \).

**Problem #12 (12 points):** Where are the following functions differentiable?

(a) \( f(z) = \tan z \).

(b) \( f(z) = 2z \).

**Solution:**

(a) \( f'(z) = \frac{1}{\cos^2 z} \) is defined everywhere that \( \cos z \neq 0 \); that is, \( z \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \).

(b) Here, \( f(z) = u + iv \) where \( u = 2x \) and \( v = -2y \). So \( u_x = 2, v_y = -2, u_y = 0, \) and \( v_x = 0 \). Thus, \( u_x \) is never equal to \( v_y \) and \( f(z) \) is nowhere differentiable.

**Problem #13 (12 points):** Show that the functions \( \text{Re} \) and \( \text{Im} z \) are nowhere differentiable.

**Solution:** \( \text{Re} = x = u + iv \), so \( u = x, v = 0, u_x = 1 \neq v_y = 0 \), so CR equations are nowhere satisfied.

\( \text{Im} z = y = u + iv \), so \( u = y, v = 0, u_y = 1 \neq v_x = 0 \), so CR equations are nowhere satisfied.

**Problem #14 (20 points):** Consider the differential equation,

\[
\frac{d^3 w}{dt^3} - kw = 0,
\]

where \( t \) is real and \( k \) is a real constant. Find the general real solution of the above equation. Write the solution in terms of real functions.

**Solution:** Look for solution of the form \( w = e^{at} \), where \( a \) is a complex constant. Then

\[
\frac{d^3 w}{dt^3} = a^3 e^{at} = a^3 w = k^3 w
\]

and so \( a^3 = k^3 \), giving

\[ a = k \quad \text{or} \quad a^2 + ka + k^2 = 0. \]

Solutions to the last equation are

\[ a_{1,2} = \frac{k}{2} \pm \frac{\sqrt{3}ki}{2}, \]

so that \( a_2 = \bar{a}_1 \). Then general complex solution of the differential equation is

\[ w = ce^{kt} + c_1 e^{a_1 t} + c_2 e^{a_2 t}, \]

where \( c, c_1 \) and \( c_2 \) are arbitrary complex constants. To make the solution real, one must restrict the constants s.t. \( c \) is real and \( c_2 = \bar{c}_1 \). Let \( c_1 = A + iB \), where \( A \) and \( B \) are real. Then

\[
w = ce^{kt} + e^{-kt/2} \left( (A + iB)e^{i\sqrt{3}kt/2} + (A - iB)e^{-i\sqrt{3}kt/2} \right) =
\]

\[
= ce^{kt} + e^{-kt/2} \left( 2A\cos(\sqrt{3}kt/2) - 2B\sin(\sqrt{3}kt/2) \right),
\]

i.e. the general real solution in terms of real functions is

\[ w = ce^{kt} + e^{-kt/2} \left( A_1 \cos(\sqrt{3}kt/2) + B_1 \sin(\sqrt{3}kt/2) \right), \]

where now \( c, A_1 \) and \( B_1 \) are arbitrary real constants.

**Problem #15 (20 points):** Establish using \( \epsilon, \delta \) arguments, that \( \lim_{z \to -i} z^3 = i \).

**Hint:** Show \( |z^3 - i| = |z + i||z^2 - iz - 1| = |z + i||z + i - 1||z + i + 1| \)

\[
= |z + i|(z + i)^2 - 3i(z + i) - 3|.
\]

Then find \( \delta = \delta(\epsilon) \) so that \( |z^3 - i| < \epsilon \) whenever \( |z + i| < \delta \).

**Solution:** Using the hint, triangle inequality, and assuming \( |z + i| < \delta \) for some \( \delta > 0 \),

\[
|z^3 - i| = |z + i||z + i + 3| < \delta(\delta^2 + 3\delta + 3).
\]

Take e.g. \( \delta < 1 \), then \( \delta(\delta^2 + 3\delta + 3) < 7\delta \). For a given \( \epsilon > 0 \), take any \( \delta \) s.t. \( \delta \leq \epsilon/7 \), then the above implies

\[ |z^3 - i| < 7\delta \leq \epsilon, \]

so \( \lim_{z \to -i} z^3 = i \).