PDE Preliminary Examination: 1/14/2015

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There are 5 problems, each worth 25 points. You are required to do 4 of them. Indicate in the table which 4 you choose—Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

1. Heat Equation

Let $Q = (0, \pi) \times (0, T)$ and $\overline{Q}$ the closure of this domain. Suppose that $u(x, t) \in C^2(Q) \times C^0(\overline{Q})$ is a solution to:

\[
\begin{align*}
    u_t(x, t) &= u_{xx}(x, t) + F(x, t), \quad (x, t) \in Q, \\
    u(0, t) &= g(t), \quad u(\pi, t) = 0, \quad t > 0, \\
    u(x, 0) &= f(x), \quad 0 \leq x \leq \pi. 
\end{align*}
\]

(a) Let $M = \max\{0, g(t), f(x) \mid (x, t) \in \overline{Q}\}$, $N = \max\{0, F(x, t) \mid (x, t) \in \overline{Q}\}$, show that $u(x, t) \leq M + tN$. (State clearly the theorems that you are using).

(b) Let $g \equiv 0$ and $F \equiv 0$. It is known that when $f'(x)$ and $f(x)$ are continuous on $[0, \pi]$ with $f(0) = f(\pi) = 0$, the above equation has a classical solution (a solution $u(x, t) \in C^2(Q) \times C^0(\overline{Q})$). Show the existence and uniqueness of a classical solution when $f$ is continuous and $f(0) = f(\pi) = 0$.

2. Fourier Series

(a) Prove the Weierstrass approximation theorem: let $f(x)$ be a $2\pi$-periodic, continuous function, then $\forall \epsilon > 0$, there exists a trigonometric polynomial $T(x)$, such that $|f(x) - T(x)| \leq \epsilon$, $\forall x \in \mathbb{R}$. (hint: construct a suitable reproducing kernel/approximation of identity).

(b) Prove Parseval’s identity: if $f(x)$ is a $2\pi$-periodic, continuous function and

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]

is its Fourier Series, then:

\[
\int_{-\pi}^{\pi} f^2(x)dx = \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).
\]

...Turn over for problem 3...
3. Method of Characteristics. The nonlinear PDE
\[ v_{tt}v_x^2 - 2v_{xt}v_tv_x + v_t^2v_{xx} = 0 \]  
(2)
is a special case of the so-called Monge-Ampère equation. In this problem, you will reduce this system to an equivalent first order equation and then solve it.

(a) Show that (2) is equivalent to:
\[ \frac{v_{tt}}{v_x} - \frac{v_{xt}v_x}{v_x^2} = \frac{v_t}{v_x} \left\{ \frac{v_{xt}}{v_x} - \frac{v_{xx}}{v_x^2} \right\} \]  
(3)
Then show that (3) can be written as an equivalent first order PDE for the new function \( u = v_t/v_x \). [Hint: we ordered the terms in (3) for a reason!]

(b) For the given initial conditions
\[ v(x,0) = 1 + 2e^{3x} \]
\[ v_t(x,0) = 4e^{3x} \]
on \(-\infty < x < \infty\), find \( u(x,t) \) for \( t > 0 \) and then find \( v(x,t) \) for \( t > 0 \).

4. Wave equation

(a) Let \( u \) be a classical solution of \( u_{tt} = c^2u_{xx} \) \((c > 0)\) on \( \mathbb{R} \times (0, \infty) \) and define
\[ E_{x,t}(s) = \frac{1}{2} \int_{x-c(t-s)}^{x+c(t-s)} \left[ u_s^2(y,s) + c^2u_y^2(y,s) \right] dy \]
for \( x \in \mathbb{R} \) and \( t \geq s > 0 \). Show that \( \frac{d}{ds}E_{x,t}(s) \leq 0 \) for \( s \in (0,t) \).

(b) For the classical solution in (a), let \( E(s) = \frac{1}{2} \int_{-\infty}^{\infty} [u_s^2(y,s) + c^2u_y^2(y,s)] dy \). Show that \( E(s) \) is monotone non-increasing, and in particular, if \( E(s_0) \) is finite then show that \( E(s) \) is finite for all \( s > s_0 \).

(c) Apply the ‘energy inequality’ from (a) to show that there is at most one classical solution to the initial value problem:
\[ u_{tt}(x,t) = c^2u_{xx}(x,t) + F(x,t), \quad x \in \mathbb{R}, \ t > 0, \]
\[ u(x,0) = f(x), \]
\[ u_t(x,0) = g(x), \]
with \( c > 0 \), such that \( u(x,t) \in C^1(\mathbb{R} \times [0, \infty)) \cap C^2(\mathbb{R} \times (0, \infty)). \)
5. Poisson’s Equation

Let $B = \{(r, \theta) | 0 \leq r < a, 0 \leq \theta < 2\pi\} \in \mathbb{R}^2$ for $a > 0$, be the open disk of radius $a$ centered at the origin, with polar coordinates $(r, \theta)$. Consider the problem

$$
\begin{align*}
\Delta u & = F(r, \theta), \quad (r, \theta) \in B, \\
u(a, \theta) & = f(\theta).
\end{align*}
$$

(a) Find a formal solution $u(r, \theta)$ that ‘solves’ (4) for $F \equiv 0$.

(b) Find conditions on $f$ that assure the formal solution $u$ obtained in part (a) is in $C^0(B)$. Give a proof of your conclusion.

(c) State and prove a version of the maximum principle (stability estimate) for (4). Apply it to prove that (4) admits at most one classical solution $u \in C^0(B) \cap C^2(B)$ for given functions $F$ and $f$. 