1. **(30 points)** For the following two ODE’s, (i) find the **general** solution (ignoring initial conditions) and (ii) the **unique** solution satisfying the initial conditions provided.

(a) (15 points)
\[ y'' - 5y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 5 \]

(b) (15 points)
\[ y'' + 2y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]

**Solution:**

Suggested grading scheme:
- 3 pts for setting up the characteristic equation.
- 3 pts for finding the correct roots
- 6 pts for finding the correct general solution (correctly dealing with repeated root in b)
- 3 pts for using the initial conditions to correctly determine A and B.

**1a.**
We can find the general solution by substituting \( e^{rt} \) into Equation 1. This yields the characteristic equation
\[ r^2 - 5r - 6 = 0, \quad (1) \]
which has roots given by
\[ r = \frac{5 \pm \sqrt{49}}{2} = 6, -1. \quad (2) \]
The general solution is then
\[ y(t) = Ae^{6t} + Be^{-t}. \quad (3) \]

Using the initial conditions, we can solve for the constants A and B. We find that
\[ y(0) = 1 \Rightarrow A + B = 2 \quad (4) \]
\[ y'(0) = 1 \Rightarrow 6A - B = 5, \quad (5) \]
which has solution \( A = B = 1 \), yielding
\[ y(t) = e^{6t} + e^{-t}. \quad (6) \]

**1b.**
The characteristic equation corresponding to Equation 1 is given by
\[ r^2 + 2r + 1 = 0, \quad (7) \]
which can be factored to yield
\[ (r + 1)^2 = 0. \quad (8) \]
We thus have a repeated root \( r = -1 \), and a general solution of the form
\[ y(t) = Ae^{-t} + Bte^{-t}. \quad (9) \]

Using our initial conditions, we can solve for A and B as follows:
\[ y(0) = 0 \Rightarrow A = 0 \quad (10) \]
\[ y'(t) = Be^{-t}(1 - t) \quad (11) \]
\[ y'(0) = 1 \Rightarrow B = 1. \quad (12) \]
The solution that passes through $y(0) = 0$, $y'(0) = 1$ is then given by

$$y(t) = te^{-t}$$

(13)
2. (30 points) Consider the harmonic oscillator system:

![Diagram of a harmonic oscillator]

Assume that the mass of the block is \( m = 2 \), the spring constant is \( k = 4 \), and that the equilibrium position of the block is \( x = 0 \). Both the damping constant \( b \) and the external force \( F(t) \) can vary. Let \( x(t) \) denote the position of the block at time \( t \). The initial conditions are \( x(0) = 1 \) and \( x'(0) = 1 \).

(a) (2 points) Write down the differential equation for \( x(t) \), governing the motion of the block, using the supplied values of \( m \) and \( k \).

(b) (8 points) Suppose the external force is zero and let \( b = 4 \). Write down the differential equation and find the solution \( x(t) \) using the supplied initial conditions. Describe the motion of the block.

(c) (14 points) Suppose no damping is present (\( b = 0 \)). An external force \( F(t) = \cos(t) \) is applied to the block. Write down the differential equation and derive the solution \( x(t) \) using the supplied initial conditions. Describe the motion of the block.

(d) (6 points) Solving for \( x(t) \) is not necessary for this question.

(I) (3 points) Suppose that \( F = 0 \) and we would like to design the damper so that the block returns as fast as possible to the equilibrium position and ceases oscillation. What must the damping constant be (or what range of values can it take) for this to be the case?

(II) (3 points) Suppose no damping is present (\( b = 0 \)) and a force \( F(t) = \cos(\sqrt{2}t) \) is applied. Describe the motion of the block in this case.

**Solution:**

(a) The differential equation for the motion of the block is \( mx'' + bx' + kx = F(t) \), so that in our case: \( 2x'' + 4x' + 4x = F(t) \) with the initial conditions \( x(0) = 1 \) and \( x'(0) = 1 \).

(b) When \( F = 0 \) and \( b = 4 \), the differential equation becomes: \( 2x'' + 4x' + 4x = 0 \). The corresponding characteristic equation is \( 2r^2 + 4r + 4 = 0 \) \( \implies r = -\frac{4 \pm \sqrt{16-32}}{4} = -1 \pm i \). Hence:

\[
x(t) = e^{-t} (C_1 \cos(t) + C_2 \sin(t))
\]

Plugging in \( x(0) = 1 \), we have \( C_1 + 0 = 1 \) \( \implies C_1 = 1 \). Next:

\[
x'(t) = -C_1 e^{-t} \cos(t) - C_1 e^{-t} \sin(t) - C_2 e^{-t} \sin(t) + C_2 e^{-t} \cos(t)
\]

Hence, the second IC gives: \( x'(0) = -C_1 + C_2 = 1 \) \( \implies C_2 = 1 + C_1 = 2 \). The solution is:

\[
x(t) = e^{-t} (\cos(t) + 2 \sin(t))
\]

The block continuously oscillates, but the amplitude of oscillations decreases towards zero with increasing time.

(c) Here, we have the equation:

\[
2x'' + 4x = \cos(t)
\]

The characteristic equation corresponding to the homogeneous equation is \( 2r^2 + 4 = 0 \) and the roots are \( r = \pm i \sqrt{2} \). The homogeneous equation solution is thus:

\[
x_h(t) = C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)
\]

Let \( x_p(t) = A \cos(t) + B \sin(t) \). Then \( x'(t) = -A \sin(t) + B \cos(t) \) and \( x''(t) = -A \cos(t) - B \sin(t) \). Plugging into the non-homogeneous equation, we get:

\[
2x''_p(t) + 4x_p(t) = -2A \cos(t) - 2B \sin(t) + 4A \cos(t) + 4B \sin(t) = \cos(t).
\]
Thus, we must have $B = 0$ and since $\cos(t)[-2A + 4A] = \cos(t)$, we have $A = \frac{1}{2}$. Hence:

$$x(t) = x_h(t) + x_p(t) = C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t) + \frac{1}{2} \cos(t)$$

The derivative is:

$$x'(t) = -C_1 \sqrt{2} \sin(\sqrt{2}t) + C_2 \sqrt{2} \cos(\sqrt{2}t) - \frac{1}{2} \sin(t)$$

so that $x(0) = C_1 + \frac{1}{2} = 1 \implies C_1 = \frac{1}{2}$. The other IC gives $x'(0) = C_2 \sqrt{2} = 1 \implies C_2 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and hence:

$$x(t) = \frac{1}{2} \cos(\sqrt{2}t) + \frac{\sqrt{2}}{2} \sin(\sqrt{2}t) + \frac{1}{2} \cos(t).$$

The block oscillates for all time and the oscillation consists of the superposition of two waves and the amplitude of oscillation is bounded for all time.

(d) (I) For fastest return to the equilibrium position, we would like the system to be critically damped. Thus, we like $b^2 - 4mk = 0 \implies b^2 = (4)(2)(4) = 32 \implies b = \sqrt{32} = \sqrt{(2)(16)} = 4\sqrt{2}$. Notice that we always have $b \geq 0$.

(II) The equation for the motion in this case is:

$$2x''(t) + 4x(t) = \cos(\sqrt{2}t)$$

Notice that $w_0 = \sqrt{\frac{k}{m}} = \sqrt{2} = w$. This is the resonant case without damping. The oscillations will increase without bound, so $|x(t)| \to \infty$ as $t \to \infty$.

Grading: (a) no partial credit. (b) 5 points for general solution, 3 points for correctly plugging in initial conditions. (c) 10 points for general solution (4 for homogeneous, 4 for particular, 2 for correct general). 4 points for correctly plugging in initial conditions. (d) (I) no partial credit, (II) mentioning resonance gives 1 point. Mentioning oscillations increase without bound gives full credit.
3. (30 points)

(a) (6 points) True or False. If \( y_1'' + 6y_1' - 16y_1 = \frac{1}{t+1}e^{-t} \) and \( y_2'' + 6y_2' - 16y_2 = \cos(3t) \), then a general solution to the differential equation
\[
y'' + 6y' - 16y = \frac{1}{t+1}e^{-t} + \cos(3t)
\]
is \( y = c_1e^{2t} + c_2e^{-8t} + y_1 + y_2 \) for some constants \( c_1 \) and \( c_2 \).

For the next questions, you may use the method of undetermined coefficients without actually solving for the coefficients (however, you may wish to solve for the coefficients if you would like to check your answer).

(b) (8 points) Find the form of a particular solution to \( y'' + 9y = \cos(3t + \pi/4) \).

(c) (8 points) Find the form of a particular solution to \( y'' + 9y = t^2e^{2t} \).

(d) (8 points) Find the form of a particular solution to \( y''' - 3y'' + 2y' - 4y = e^{2t} \)

\[ \text{Solution:} \]

(a) True

(b) either \( t(A \cos(3t + \pi/4) + B \sin(3t + \pi/4)) \) or \( t(A \cos(3t) + B \sin(3t)) \) [or \( t(Ae^{3it} + Be^{-3it}) \)]

(c) \( e^{2t} (At^2 + Bt + C) \)

(d) \( Ae^{2t} \); you should check that \( r = 2 \) is not a root of \( r^3 - 3r^2 + 2r - 4 \) since otherwise you would need \( Ae^{2t} \) (it isn’t, since \( 2^3 - 3 \cdot 2^2 + 2 \cdot 2 - 4 = 8 - 12 + 4 - 4 = -4 \neq 0 \)).
4. (30 points)
   (a) (15 points) Find the Laplace transform $G(s)$ of
   \[ g(t) = e^t \sin(t). \]
   (b) (15 points) Find $y(t)$ if its Laplace transform is
   \[ Y(s) = \frac{s - 1}{s^2 - 6s + 10}. \]

Solution:
(a) Using the property $\mathcal{L}(tf(t)) = -F'(s)$ with $F(s) = \mathcal{L}(\sin(t)) = \frac{1}{s^2 + 1}$, we get $\mathcal{L}(t \sin(t)) = -F'(s) = \frac{2s}{(s^2 + 1)^2}$. Now using the property $\mathcal{L}(e^t g(t)) = G(s - 1)$ we obtain
\[ \mathcal{L}(e^t \sin(t)) = \frac{2(s - 1)}{(s - 1)^2 + 1}. \]
(b) The roots of $s^2 - 6s + 10 = 0$ are complex, so we complete the square in the denominator,
\[ Y(s) = \frac{s - 1}{s^2 - 6s + 10} = \frac{s - 1}{(s - 3)^2 + 1}. \]
We write everything in terms of $s - 3$,
\[ Y(s) = \frac{s - 3 + 2}{(s - 3)^2 + 1} = \left[ \frac{s - 3}{(s - 3)^2 + 1} + \frac{2}{(s - 3)^2 + 1} \right] = \mathcal{L}[e^{3t} \cos(t) + 2e^{3t} \sin(t)]. \]
5. (30 points) Consider the following IVP

\[ y'' - 3y' + 2y = -2e^{-2t} \quad y(0) = 0, \quad y'(0) = 1. \]

(a) (3 points) Given the Laplace transform \( \mathcal{L}[y(t)] = Y(s) \), determine the form for \( \mathcal{L}[y'(t)] \).

(b) (3 points) Given the Laplace transform \( \mathcal{L}[y(t)] = Y(s) \), determine the form for \( \mathcal{L}[y''(t)] \).

(c) (3 points) Compute the Laplace transform of \(-2e^{-2t}\).

(d) (6 points) Using your answers from (a)-(c), find \( Y(s) \) the Laplace transform to the solution of the IVP.

(e) (15 points) Find the solution to the IVP by taking the inverse Laplace transform of \( Y(s) \).

\[ \text{Solution:} \]

Note

(a) \( \mathcal{L}[y''(t)] = s^2Y(s) - s.0 - 1 = s^2Y(s) - 1 \)

(b) \( \mathcal{L}[y'(t)] = sY(s) - s.0 = sY(s) \)

(c) \( \mathcal{L}[-2e^{-2t}] = -\frac{2}{s + 2} \)

(d) Laplace transforming the IVP gives

\[
(s - 2)(s - 1)Y(s) - 1 = -\frac{2}{s + 2} \\
(s - 2)(s - 1)Y(s) = \frac{s}{s + 2} \\
Y(s) = \frac{s}{(s + 2)(s - 1)(s - 2)}
\]

(e) By partial fraction

\[ Y(s) = \frac{A}{s - 2} + \frac{B}{s - 1} + \frac{C}{s + 2} \quad \Rightarrow \quad A(s - 1)(s + 2) + B(s - 2)(s + 2) + C(s - 2)(s - 1) = s \]

Setting \( s = 2 \) gives \( A = 1/2 \); \( s = 1 \) gives \( B = -1/3 \); \( s = -2 \) gives \( C = -1/6 \). Therefore

\[ Y(s) = \frac{1}{2} \left( \frac{1}{s - 2} \right) - \frac{1}{3} \left( \frac{1}{s - 1} \right) - \frac{1}{6} \left( \frac{1}{s + 2} \right) \]

By inverse transform from the lookup table we have

\[ y(t) = \frac{1}{2}e^{2t} - \frac{1}{3}e^t - \frac{1}{6}e^{-2t} \]
### Table 8.1.1 Short table of Laplace transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $1$</td>
<td>$\frac{1}{s}$ $s &gt; 0$</td>
</tr>
<tr>
<td>(ii) $t^n$</td>
<td>$\frac{n!}{s^{n+1}}$ $s &gt; 0, n$ a positive integer</td>
</tr>
<tr>
<td>(iii) $e^{at}$</td>
<td>$\frac{1}{s-a}$ $s &gt; a$</td>
</tr>
<tr>
<td>(iv) $t^n e^{at}$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$ $s &gt; a, n$ a positive integer</td>
</tr>
<tr>
<td>(v) $\sin(bt)$</td>
<td>$\frac{b}{s^2 + b^2}$ $s &gt; 0$</td>
</tr>
<tr>
<td>(vi) $\cos(bt)$</td>
<td>$\frac{s}{s^2 + b^2}$ $s &gt; 0$</td>
</tr>
<tr>
<td>(vii) $e^{at} \sin(bt)$</td>
<td>$\frac{(s-a)^2 + b^2}{s-a}$ $s &gt; a$</td>
</tr>
<tr>
<td>(viii) $e^{at} \cos(bt)$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$ $s &gt; a$</td>
</tr>
<tr>
<td>(ix) $\sinh(bt)$</td>
<td>$\frac{s}{s^2 - b^2}$ $s &gt;</td>
</tr>
<tr>
<td>(x) $\cosh(bt)$</td>
<td>$\frac{s}{s^2 - b^2}$ $s &gt;</td>
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