Problem 1: (20 points) A competition model for the number of “A” students that Ed (E) and Tony (T) have is the system.

\[ E' = E(-8 + 2T) \]
\[ T' = T(3 - E) \]

(a) Find the equations of all nullclines. Sketch all nullclines. Use E for the horizontal axis. Failing to do so will result in a loss of points. Make your sketch large, because you will add to it in the rest of this problem.

(b) On your sketch from (a), indicate the equilibrium point(s), giving their exact coordinates. Describe physically what each equilibrium point represents.

(c) On your sketch from (a), indicate with arrows the direction of movement along the nullclines.

(d) Describe the long-term behavior of the two populations if (i) \( E(0) = 0 \) and \( T(0) = 2 \), and (ii) \( E(0) = 3 \) and \( T(0) = 3 \)? On your sketch from (a), indicate these solution trajectories.

Solution 1:

(a) \( E' = 0 \) nullclines are: \( E' = 0 \Leftrightarrow E(-8 + 2T) \Rightarrow E = 0 \) and \( T = 4 \) (red dashed lines in the plot)

\[ T' = 0 \] nullclines are: \( T' = 0 \Leftrightarrow T(3 - E) \Rightarrow T = 0 \) and \( E = 3 \) (blue solid lines in the plot)

(b) Intersections of \( E' = 0 \) nullclines with \( T' = 0 \) nullclines give the equilibrium points. Since the nullclines are simply vertical and horizontal lines, it is okay for little/no work to be shown in determining the coordinates of the equilibria, as long as the plot from (a) is fine.
⇒ Equilibrium points at $(E, T) = (0, 0)$ and $(3, 4)$. The first represents no students with As. The second represents Tony has 4 students with As, Ed has 3.

(c) (see plot)
(d) (i) If $E(0) = 0$ and $T(0) = 2$, then eventually $T = \infty$ and $E = 0$
(ii) If $E(0) = 3$ and $T(0) = 3$, then eventually solutions revolve clockwise around the equilibrium at $(3, 4)$

**Note:** It is also acceptable for students to show this as a spiral into or out from the equilibrium at $(3, 4)$, since the nullcline analysis does not reveal which of these 3 cases is at hand, as long as the statement is consistent with the trajectory sketch.

**Problem 2:** (20 points) Consider the system of equations

\[
\begin{align*}
  x_1 + 5x_3 &= 3 \\
  4x_1 + 2x_2 &= 2 \\
  3x_1 + x_2 + cx_3 &= 4
\end{align*}
\]

where $c$ is an unknown constant until specified.

(a) Write the system of equations as one matrix equation $Ax = b$. **BE CAREFUL! Doing this wrong can lead to a horrendous loss of points.**

(b) Find the determinant of $A$ or explain why it cannot be calculated.

(c) Determine, if possible, all values of $c$ that make $A$ invertible. You may not use any calculations other than your result from (b).

(d) Choose $c = 5$ and find all solutions to this system.

(e) Find one solution to $Ax = 0$ other than $x = 0$. **Hint:** Use your solution from (d).

**Solution 2:**

(a)

\[
A = \begin{pmatrix} 1 & 0 & 5 \\ 4 & 2 & 0 \\ 3 & 1 & c \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}.
\]

(b) $\det(A) = 2c - 10$.

(c) $A$ invertible means $\det(A) \neq 0$. So, we would require $c \neq 5$.

(d) Row reducing gives

\[
x = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 10 \\ 1 \end{pmatrix}
\]

for any choice of $x_3$.

(e) The second part of the answer from above is all possible homogeneous solutions. So, in particular, we know

\[
x = \begin{pmatrix} -5 \\ 10 \\ 1 \end{pmatrix}
\]

solves $Ax = 0$. 
Problem 3: (20 points) Parts (a) and (b) are the only related parts. You must justify all responses.

(a) Find the inverse of

\[
A = \begin{pmatrix}
0 & 3 & 0 \\
0 & 0 & 4 \\
9 & 0 & 0
\end{pmatrix}.
\]

(b) Without pointing to your answer from (a) or using determinants, show that \( A \) is invertible.

(c) Suppose that \( A, B, C, \) and \( D \) are four \( 2 \times 2 \) matrices with

\[
ABCD = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}.
\]

Show that \( A, B, C, D \) are invertible and find a formula for \( A^{-1} \).

(d) Let \( A \) be a \( 50 \times 50 \) matrix with determinant 2360. Now suppose that you swap three pairs of rows and call the result \( B \). What is \( \text{rank}(B) \)?

Solution 3:

(a) Basic calculations give us

\[
A^{-1} = \begin{pmatrix}
0 & 0 & 1/9 \\
1/3 & 0 & 0 \\
0 & 1/4 & 0
\end{pmatrix}.
\]

(b) The columns of \( A \) are clearly linearly independent.

(c) Taking the determinant of both sides, we get \( \text{det}(A)\text{det}(B)\text{det}(C)\text{det}(D) = -2 \), which means \( \text{det}(A) \neq 0 \) and the same for \( B, C, \) and \( D \). Hence, they must all be invertible. Left multiply by \( A^{-1} \) and then right multiply by the inverse of RHS matrix:

\[
A^{-1} = BCD \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}^{-1} = BCD \begin{pmatrix}
-2 & 1 \\
3/2 & -1/2
\end{pmatrix}
\]

(d) \( \text{det}(B) = (-1)^3 \text{det}(A) = -2360 \neq 0 \) so \( \text{rank}(B) = \text{rank}(A) = 50 \).
**Problem 4:** (20 points) The following parts are not related to each other.

(a) The matrix

\[
A = \begin{pmatrix}
1 & 2 & 3 & 1 & 1 & 1 \\
4 & 5 & 6 & 1 & 1 & 1 \\
7 & 8 & 9 & 0 & 0 & 0
\end{pmatrix}
\]

is row equivalent to

\[
B = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

Determine \(\text{rank}(A)\) with justification. What does this tell you about the invertibility of \(A\)?

(b) Determine, with proof, if \(S = \{1, 1 - x + x^2, 2 - x^2, 4 - 7x + 21x^2\}\) is a basis for \(\mathbb{P}_2\).

**Solution 4:**

(a) The first, second, and fourth columns of \(B\) contain pivots, so there are three pivot columns in \(A\). Hence, \(\text{rank}(A) = 3\). This tells us NOTHING about invertibility because \(A\) is not square.

(b) The set has four elements from a space of dimension 3. This set cannot be linearly independent and thus \(S\) is not a basis.
Problem 5: (20 points) Grab Bag. Justification is required only for parts (c), (d), and (e).

(a) If $|A| = 2$, $|B| = 5$ and $|C| = 3$, then assuming the matrices are all the correct size for the following multiplication to be defined, what is $|(AB)^{-1}C^T C|$? If it cannot be calculated from the given information, say so.

(b) What is the determinant of
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 5 & 6 & 7 \\
0 & 0 & 8 & 9 \\
0 & 0 & 0 & 10
\end{pmatrix}.
\]

(c) Give an example of a basis for $\mathbb{M}_{2 \times 2}$, the space of $2 \times 2$ matrices. Justify, but feel free to use intuition.

(d) Give an example of a set of vectors in $\mathbb{R}^3$ that is linearly independent but does not form a basis for $\mathbb{R}^3$.

(e) Let $W$ be the set of all $2 \times 2$ matrices with determinant 0. Is $W$ a vector space?

(f) (0 pts) In the vector space of all delicious foods, what do you think are the basis vectors?

Solution 5:

(a) $|(AB)^{-1}C^T C| = |B^{-1}A^{-1}C^T C| = |B^{-1}||A^{-1}||C^T||C| = \frac{1}{|B|} \frac{1}{|A|}|C||C| = \frac{1}{5} \frac{1}{2} 3^2 = \frac{9}{10}$

(b) $400$. The matrix is triangular, so the determinant is the product of entries on the main diagonal.

(c) The standard basis:
\[
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The intuition here is that each basis element picks out an entry for any element of $\mathbb{M}_{2 \times 2}$.

(d) Let $S = \{(1,0,0)\}$. A set with one vector is automatically linearly independent. However, $\text{span}(S)$ cannot be all of $\mathbb{R}^3$ because we are missing, for example, $(0,1,0) \in \mathbb{R}^3$ but $(0,1,0) \notin \text{span}(S)$.

(e) No

Consider $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, which are both elements of $W$. $M_1 + M_2 = I_{2 \times 2}$ and thus $\det(M_1 + M_2) = 1$, so $M_1 + M_2 \notin W$

(f) Obviously, they are bread, sauce and cheese. And maybe high fructose corn syrup.