Problem 1: (20 points)

(a) Find all solutions to
\[
\begin{align*}
    \frac{dy}{dt} &= (3 - 2y) \cos(t) \\
    y(0) &= 1
\end{align*}
\]

(b) The ODE \(\frac{1}{t} z' + t^2 z = 0\) has solution: \(z_h(t) = C e^{-t^4/4}\). Use this fact and Variation of Parameters to solve \(\frac{1}{t} z' + t^2 z = e^{-t^4/4}\).

Solution 1:

(a) First, we see that there is an equilibrium solution \(y_e(t) = 3/2\), which DOES NOT satisfy the initial condition. Now using Separation of Variables, we see that
\[
\int \frac{dy}{3 - 2y} = \int \cos(t) \, dt \Rightarrow -\frac{1}{2} \ln(3 - 2y) = \sin t + C
\]
\[
\Rightarrow y(t) = \frac{1}{2} \left(3 - C e^{-2 \sin(t)}\right).
\]
Using our initial condition we find that \(C = 1\) which gives a final answer of:
\[
y(t) = \frac{1}{2} \left(3 - e^{-2 \sin(t)}\right).
\]

(b) Variation of Parameters gives us a guess for a solution to the non homogeneous ODE based on the homogeneous solution. Let that guess be \(z_p(t) = v(t) e^{-t^4/4}\), where \(v(t)\) is to be determined. We can see that
\[
z_p' = \frac{1}{t} \left(v'(t) - t^3 v(t)\right) e^{-t^4/4}
\]
and thus replacing \(z_p\) for \(z\) in the ODE yields
\[
\frac{1}{t} \left(v'(t) - t^3 v(t)\right) e^{-t^4/4} + t^2 v(t) e^{-t^4/4} = e^{-t^4/4} \Rightarrow v' = t.
\]
This gives us a particular solution of \(z_p(t) = v(t) e^{-t^4/4} = \frac{t^2}{2} e^{-t^4/4}\), so that in total our solution is
\[
z(t) = z_h(t) + z_p(t) = (C + \frac{t^2}{2}) e^{-t^4/4}.
\]
Problem 2: (20 points)

The depletion of an original sample of uranium ore of 30 kg is modeled by the differential equation \( y' = -ky \), with \( y(0) = 30 \). Use units of billions of years for time, \( t \). Failing to do so will result in a loss of points.

Parts (a)-(c) are related; part (d) is unrelated.

(a) Find the solution to the IVP \( \frac{dy}{dt} = -ky \), \( y(0) = 30 \). Show all work. Solving “by inspection” will be awarded minimal credit.

(b) According to the above review, at \( t_{1/2} = 4.47 \), the untouched container of uranium ore belonging to the Procrastinating Evil Scientist was half empty. What is \( k \)?

(c) In order to become the superhero Radioactive Man, Norm Al Guy needs to be exposed to at least 10 kg of uranium ore. At what time will it no longer be possible for Norm to obtain superpowers?

(d) When studying anything except differential equations, your happiness \( H(t) \) decreases at a rate proportional to your current happiness. But when you eat a snack, your happiness increases by an amount \( S \), and when you watch an internet video, your happiness increases by an amount \( V \). Use units of hours for time, \( t \), and suppose you eat a snack every 2 hours and watch an internet video every hour. What differential equation models your happiness while studying subjects besides differential equations? \textbf{DO NOT SOLVE IT!}

Solution 2:

(a) We use separation of variables:

\[
\frac{dy}{dt} = -ky \\
\frac{dy}{y} = -k \, dt \\
\int \frac{dy}{y} = - \int k \, dt \\
\ln|y| = -kt + C \\
e^{\ln|y|} = y(t) = e^{-kt+C} = e^{-kt}e^C = Ae^{-kt}
\]

And apply the initial condition to determine \( C \): \( y(0) = 30 \Rightarrow Ae^{-k(0)} \Rightarrow A = 30 \).

Therefore the solution is \( y(t) = 30e^{-kt} \).
(b) At $t = t_{1/2}$, the half-life, $y = \frac{1}{2} y_0 = \frac{1}{2} (30) = 15$. Therefore, we have:

$$y(4.47) = 15 \equiv 30e^{-k(4.47)}$$

$$\frac{1}{2} = e^{-4.47k}$$

$$\ln \left( \frac{1}{2} \right) = \ln \left( e^{-4.47k} \right) = -4.47k$$

$$k = \frac{\ln \left( \frac{1}{2} \right)}{-4.47}$$

Alternatively, $k = \frac{\ln 2}{4.47}$

(c) If we let $T$ equal the time when $y(T) = 10$, then $T$ is the last time Norm can obtain superpowers. So solve for $T$:

$$y(T) = 10 \equiv 30e^{-kT} \quad \text{(we'll plug in } k \text{ from part (b) at the end)}$$

$$\frac{1}{3} = e^{-kT}$$

$$\ln \left( \frac{1}{3} \right) = \ln \left( e^{-kT} \right)$$

$$\ln \left( \frac{1}{3} \right) = -kT$$

$$T = -\frac{\ln \left( \frac{1}{3} \right)}{k}$$

$$T = \frac{\ln 3}{k}$$

$$T = \frac{\ln 3}{(\frac{\ln 2}{4.47})}$$

$$T = 4.47 \frac{\ln 3}{\ln 2} \quad \text{or} \quad T = 4.47 \frac{\ln 1/3}{\ln 1/2}$$

(d) $\frac{dH}{dt} = -kH + \frac{1}{2}S + V$
Problem 3: (20 points) Sam and Ella’s, a famous seafood restaurant, makes a special sauce by mixing spice Xanthippe with water. Starting with a 5L tank containing 1L of water and 1 kg of spice X, a mix with 2 kg/L of spice X is poured into the tank at a rate of 3 L/min. The contents of the tank are well mixed and leave the tank at a rate of 3 L/min. Professor Chaos, bringer of all doom, clogs the tank so that the sauce leaves the tank at 2L/min.

(a) Set up the IVP associated with this problem. You may assume that $t = 0$ is when the machine is turned on AFTER Professor Chaos’ sabotage.

(b) Solve this IVP using the Integrating Factor Method. Minimal credit will be awarded for simply using a formula that yields the result.

(c) Describe physically what happens to the restaurant if this sabotage goes unnoticed.

Solution 3:

(a) Let $x(t)$, measured in kg, denote how much Spice X is in the tank at time $t$, measured in min.

\[
\frac{dx}{dt} = 6 - \frac{2x}{1 + t}
\]

$x(0) = 1$

$t \in [0, \infty)$

(b) Rearrange the ODE as

\[
\frac{dx}{dt} + \frac{2x}{1 + t} = 6.
\]

The integrating factor is $\mu(t) = (1 + t)^2$. Multiply the ODE on both sides by $\mu(t)$ and recognize the product rule to get:

\[
\frac{d}{dt} \left( (1 + t)^2 x(t) \right) = 6(1 + t)^2 \implies (1 + t)^2 x(t) = 2(1 + t)^3 + C
\]

\[
\implies x(t) = \frac{2(1 + t)^3 + C}{(1 + t)^2}.
\]

Now require that $x(0) = 1$:

\[
1 = 2 + C \implies C = -1.
\]

The solution to the IVP is thus

\[
x(t) = \frac{2(1 + t)^3 - 1}{(1 + t)^2}.
\]

Alternatively, if you FOIL the $\int (1 + t)^2 dt$ integral, you’ll find

\[
x(t) = \frac{2t^3 + 6t^2 + 6t + 1}{(1 + t)^2}.
\]

(c) The tank will overflow starting from $t = 4$ minutes so that in the long run, Sam and Ella’s will be flooded with the mixture. We can see this by either noting that the amount of water is $W(t) = 1 + t$ or by checking that

\[
\lim_{t \to \infty} x(t) = \infty.
\]
Problem 4: (20 points) Consider the differential equation \( \frac{dy}{dt} = (y^2 - y - 2)(t + 2) \). The direction field for this equation is shown below.

(a) Classify the ODE by stating the order and discussing linearity and separability.

(b) Using the differential equation, find all equilibrium solutions. Discuss stability.

(c) What can you say about the existence and uniqueness of solutions for any given initial conditions, \((t_0, y_0)\)? Fully justify your answer and state any theorem(s) you use.

(d) Using initial conditions (i) \( y(0) = 0 \) and (ii) \( y(1) = 2 \), describe the end behavior of solutions.

Solution 4:

(a) \( \frac{dy}{dt} = (y^2 - y - 2)(t + 2) \) \( \iff \) First order, nonlinear, separable

(b) Equilibrium points are where \( \frac{dy}{dt} = 0 \), so

\[
0 = (y^2 - y - 2)(t + 2) \\
0 = (y - 2)(y + 1)(t + 2)
\]

\( y = -1, 2 \)

From the direction field, \( y = -1 \) is stable because the direction field arrows converge there as \( t \to \infty \). \( y = 2 \) is unstable because the direction field arrows diverge at \( t \to \infty \).

(c) We use Picard’s Theorem:

- \( \frac{dy}{dt} = f(t, y) = (y^2 - y - 2)(t + 2) \) is continuous for all \( t \) and \( y \), so for any initial conditions, at least one solution exists.
- Differentiate with respect to \( y \) to check uniqueness: \( f_y = (2y - 1)(t + 2) \) is continuous for all \( t \) and \( y \) as well, therefore a unique solution exists for any initial conditions.

(d) (i) From \( y(0) = 0 \), the direction field shows that solutions initiating from \((0, 0)\) tend toward the equilibrium at \( y = -1 \).

(ii) From part (b), we know \( y = 2 \) is an equilibrium. Therefore, solutions which start there stay there, so solutions stay at \( y = 2 \).
Problem 5: (20 points) **Grab bag.** Justification is required only for (c) and (d).

(a) **True or False:** Given the IVP \( y' = f(t, y) \), \( y(t_0) = y_0 \), if \( f \) is not continuous at \((t_0, y_0)\), then no solutions exist.

(b) **True or False:** If \( L \) is a linear operator, \( y \) and \( z \) are functions, and \( \alpha \) and \( \beta \) are real numbers, then \( L(\alpha y + \beta z) = \alpha L(y) + \beta L(z) \).

(c) Is every first order autonomous ODE separable?

(d) If you were to use Euler’s Method on the IVP \( y' = 1 + y^{2360}, \ y(1) = 1 \), will the approximation be an underestimate, an overestimate, exact, or none of the above?

**Solution 5:**

(a) **False**

The statement is the converse of Picard’s Theorem, which is not true!

(b) **True**

This is an equivalent statement to linearity of \( L \).

(c) **YES!**

Every first order autonomous ODE can be written in the form \( y' = f(y) \), which is clearly separable.

(d) **The approximation is an underestimate.**

It’s easy to use a direction field to argue that the solution will be concave up in the first quadrant (a more formal argument would be to look at \( y''(t) \), which we can get from the Chain Rule). The true solution MUST lie in the first quadrant because the initial condition starts there and \( y' > 0 \). This means that the Euler iterates will approximate a function that is concave up. Since Euler’s Method uses a linear approximation, the iterates must be less than the actual solution.