Problem 1: (20 points) A competition model for the number of “A” students that Ed (E) and Tony (T) have is the system.

\[ E' = E(-8 + 2T) \]
\[ T' = T(3 - E) \]

(a) Find the equations of all nullclines. Sketch all nullclines. Use E for the horizontal axis. Failing to do so will result in a loss of points. Make your sketch large, because you will add to it in the rest of this problem.

(b) On your sketch from (a), indicate the equilibrium point(s), giving their exact coordinates. Describe physically what each equilibrium point represents.

(c) On your sketch from (a), indicate with arrows the direction of movement along the nullclines.

(d) Describe the long-term behavior of the two populations if (i) \( E(0) = 0 \) and \( T(0) = 2 \), and (ii) \( E(0) = 3 \) and \( T(0) = 3 \)? On your sketch from (a), indicate these solution trajectories.

Problem 2: (20 points) Consider the system of equations

\[ x_1 + 5x_3 = 3 \]
\[ 4x_1 + 2x_2 = 2 \]
\[ 3x_1 + x_2 + cx_3 = 4 \]

where \( c \) is an unknown constant until specified.

(a) Write the system of equations as one matrix equation \( Ax = b \). BE CAREFUL! Doing this wrong can lead to a horrendous loss of points.

(b) Find the determinant of \( A \) or explain why it cannot be calculated.

(c) Determine, if possible, all values of \( c \) that make \( A \) invertible. You may not use any calculations other than your result from (b).

(d) Choose \( c = 5 \) and find all solutions to this system.

(e) Find one solution to \( Ax = 0 \) other than \( x = 0 \). Hint: Use your solution from (d).

Problem 3: (20 points) Parts (a) and (b) are the only related parts. You must justify all responses.

(a) Find the inverse of

\[ A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 9 & 0 & 0 \end{pmatrix} \]

(b) Without pointing to your answer from (a) or using determinants, show that \( A \) is invertible.
(c) Suppose that $A, B, C,$ and $D$ are four $2 \times 2$ matrices with

$ABCD = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Show that $A, B, C, D$ are invertible and find a formula for $A^{-1}$.

(d) Let $A$ be a $50 \times 50$ matrix with determinant 2360. Now suppose that you swap three pairs of rows and call the result $B$. What is rank($B$)?

Problem 4: (20 points) The following parts are not related to each other.

(a) The matrix

$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 4 & 5 & 6 & 1 & 1 \\ 7 & 8 & 9 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

is row equivalent to

$B = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

Determine rank($A$) with justification. What does this tell you about the invertibility of $A$?

(b) Determine, with proof, if $S = \{1, 1 - x + x^2, 2 - x^2, 4 - 7x + 21x^2\}$ is a basis for $\mathbb{P}_2$.

Problem 5: (20 points) Grab Bag. Justification is required only for parts (c), (d), and (e).

(a) If $|A| = 2$, $|B| = 5$ and $|C| = 3$, then assuming the matrices are all the correct size for the following multiplication to be defined, what is $|(AB)^{-1}C^TC|$? If it cannot be calculated from the given information, say so.

(b) What is the determinant of

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$?

(c) Give an example of a basis for $\mathbb{M}_{2 \times 2}$, the space of $2 \times 2$ matrices. Justify, but feel free to use intuition.

(d) Give an example of a set of vectors in $\mathbb{R}^3$ that is linearly independent but does not form a basis for $\mathbb{R}^3$.

(e) Let $W$ be the set of all $2 \times 2$ matrices with determinant 0. Is $W$ a vector space?

(f) (0 pts) In the vector space of all delicious foods, what do you think are the basis vectors?

(A) Bacon-wrapped french fries, anyone?  
(b) or maybe some hot dog-stuffed crust pizza!