Problem 1:

(a) (2 points) Write the characteristic equation for $x'' + bx' + 4x = 0$ and solve its roots $r_{1,2}$ for general $b$ using the quadratic formula.

(b) (9 points) For which values of $b$ is the system over-damped, critically-damped, and under-damped? For each case, sketch and describe: (i) the general behavior of $x(t)$ versus $t$ for each situation, and (ii) the corresponding phase plot depicting $x'(t)$ versus $x(t)$.

(c) (5 points) Let $b = 0$ and use your above expression for the roots to derive the general solution in terms of sine and cosine functions. [Hint: use Euler’s expansion for $e^{i\theta}$ to convert the exponentials to the trigonometric functions].

(d) (4 points) Solve the IVP for the initial conditions $x(0) = 1$ and $x'(0) = 4$.

Problem 2:

(a) (5 points) Given the second-order differential equation $x'' + 5x' + 4x = 0$, convert the equation to a system of two first order differential equations and write the system in matrix form, $\frac{d}{dt} (\vec{y}) = A\vec{y}$.

(b) (6 points) Find and plot the null clines. Be sure to indicate the direction field for all the regions of the phase portrait as well as on the null clines.

(c) (5 points) Find the general solution and draw an example trajectory in the phase portrait drawn in (b).

(d) (4 points) Find and classify the stability of all equilibria.

Problem 3:

Let $L(y)$ denote the linear operator $L(y) = y'' - 2y' + y$.

(a) (8 points) For each following forcing functions $f(t)$, write down a suitable form for $y_p$ to solve $L(y) = f(t)$ using the method of undetermined coefficients. [You do not need to solve the coefficients.]

(i) $f(t) = 2 + t^3$
(ii) $f(t) = e^t$
(iii) $f(t) = \cos(t) + \sin(2t)$
(iv) $f(t) = e^{2t} + t \cos(t)$.

(b) (8 points) Let $g(t) = 100 \sin(3t) + 5t$. Solve $y_p$ for $L(y) = g(t)$ using the method of undetermined coefficients (i.e., evaluate all coefficients).
(c) (4 points) Using \(g(t)\) and your \(y_p\) from part (b), find the general solution to the initial value problem (IVP) \(L(y) = g(t)\) with initial conditions \(y(0) = 16\) and \(y'(0) = 0\).

**Problem 4:**

Let \(L(y) = t^2y'' - 2ty' + 2y\), \(y_1 = t\), and \(y_2 = t^2\).

(a) (4 points) Show that \(y_1\) and \(y_2\) are solutions to the corresponding homogeneous equation, \(L(y_1) = L(y_2) = 0\).

(b) (4 points) Prove that \(\{y_1, y_2\}\) are a basis for the solution space to the homogeneous equation, and thus any solution may be written as \(y_h = C_1y_1 + C_2y_2\).

(c) (8 points) Find \(y_p\) for \(L(y) = t^3\sin(t)\) using variation of parameters. [Hint: you may need to use \(\int udv = uv - \int vdu\) for integration.]

(d) (4 points) Use your results from (c) to solve the initial value problem \(L(y) = t^3\sin(t)\) using that at time \(t_0 = 1\) the initial conditions are \(y(1) = -\sin(1)\) and \(y'(1) = -\sin(1)\). Write the full solution \(y(t)\) to the IVP.

**Problem 5:**

Let \(u'' + bu' + 100u = A\cos(\omega f t)\).

(a) (4 points) Now consider the forced system with \(b = 0\) and \(A = 40\). Write the general solution \(x_G\) for (i) when \(\omega f\) does not equal the resonant frequency and (ii) when it does. [You do not need to derive the solutions, but simplify as much as possible.]

Does there exist a forcing frequency \(\omega f\) for which the general solution \(x_G = x_h + x_p\) becomes unbounded?

(b) (6 points) Use your solution from part (a) for \(\omega f = \sqrt{60}\) to solve the IVP for \(u(0) = 1\) and \(u'(0) = 50\).

(c) (5 points) Now consider the damped system with \(b = 16\), \(A = 1\), and \(\omega f = 1\). Find the general solution. You do not need to derive the particular solution \(y_p\), but simplify as much as possible. [Hint: try bringing the denominator of the quadratic equation into the square root to find the roots of characteristic polynomial.]

(d) (5 points) Use your solution from part (c) to solve the IVP with \(u(0) = u'(0) = 0\). [You may express the constants in terms of variables defined for your particular solution.]