Solution: Problem 1

(a) 3rd order, linear, non-homogeneous, variable coefficient
(b) 1st order, linear, homogeneous, variable coefficient
(c) 2nd order, non-linear
(d) 2nd order, linear, non-homogeneous, constant coefficient

Solution: Problem 2

(a) The integrating factor is:
$$
\mu = e^{\int (3t^2 + \frac{1}{t}) dt} = e^{t^3 + \ln t} = te^{t^3}
$$

(b) The general solution to the differential equation is:
$$
y' + (3t^2 + \frac{1}{t})te^{t^3} y = t^2 e^{t^3}
$$
$$
\frac{d}{dt}[te^{t^3} y] = t^2 e^{t^3}
$$
$$
te^{t^3} y = \int t^2 e^{t^3} dt = \frac{1}{3} e^{t^3} + C
$$
$$
y = \frac{1}{3t} + \frac{Ce^{-t^3}}{t}
$$

(c) The solution to the initial value problem is:
$$
y(1) = \frac{4}{3} = \frac{1}{3} + \frac{Ce^{-1}}{1} \implies C = e
$$
$$
y(t) = \frac{1}{t} \left( \frac{1}{3} + e^{(1-t^3)} \right)
$$

Solution: Problem 3

(a) The doubling time for the colony is one day. Therefore, we know that
$$
y_0 e^{k-1} = 2y_0
$$
$$
e^k = 2
$$
$$
\therefore \quad k = \ln(2)
$$

(b) The differential equation is
$$
\frac{dy}{dt} = \ln(2)y - 24
$$
(c) The quickest way to solve this equation is via separation of variables

\[ \frac{dy}{\ln(2)y - 24} = dt \]

\[ \int \frac{dy}{\ln(2)y - 24} = \int dt \]

\[ \frac{\ln|\ln(2)y - 24|}{\ln(2)} = t + C_1 \]

\[ \ln(2)y - 24 = C_2e^{\ln(2)t} \]

\[ y(t) = C_3e^{\ln(2)t} + \frac{24}{\ln(2)} \]

\[ y(t) = C_32^t + \frac{24}{\ln(2)} \]

for unknown constant $C_3$. And, from the initial condition we obtain

\[ y(0) = 40 = C_32^0 + \frac{24}{\ln(2)} \]

\[ C_3 = 40 - \frac{24}{\ln(2)} \]

and thus the full solution is

\[ y(t) = \left(40 - \frac{24}{\ln(2)}\right)2^t + \frac{24}{\ln(2)} . \]

**Solution: Problem 4**

T(0) = 140 F, T(1) = 136 F

\[ \frac{dT}{dt} = k(60 - T) \]

\[ \frac{dT}{dt} + kT = 60k \]

μ = $e^{kt}$

\[ \frac{d}{dt}[e^{kt}T] = 60ke^{kt} \]

$e^{kt}T = 60e^{kt} + C$

T = 60 + Ce^{-kt}

T(0) = 140 = 60 + C \implies C = 80

\[ T = 60 + 80e^{-kt} \]

T(1) = 136 = 60 + 80e^{-k}

\[ - \ln\left(\frac{76}{80}\right) = k = - \ln\left(\frac{19}{20}\right) \]

Now, 80 = 60 + 80$e^{\ln\left(\frac{19}{20}\right)t}$

\[ \ln(1/4)/\ln(19/20) = t \approx 27\text{min} \]

**Solution: Problem 5**

(a) Equilibria occur when $y' = 0$. In this case, at $y = 1$ and $y = 2$.

(b) The equilibrium solution $y = 2$ is unstable because nearby solutions tend away from $y = 2$, and the solution $y = 1$ is stable because nearby solutions tend towards $y = 1$ as $t$ increases.
(c) The differential equation undergoes a qualitative change when $a = 1$. In particular, the system always has equilibria at $y = 1$ and $y = a$. When $a < 1$, $y = a$ is a stable equilibrium and $y = 1$ is unstable. When $a > 1$, $y = a$ is unstable and $y = 1$ is stable. When $a = 1$, the equilibria coincide and so the system has a single semi-stable equilibrium.