Problem 1: (30 points) Method of undetermined coefficients.

(a) (20 points) Solve the following problem

\[ y'' + 2y' = 4t + 5 \cos t \]

(b) (10 points) Consider the following ODE

\[ y'' + 2y' + 2y = e^{-t} \sin t. \]

Without solving, write down the form of the particular solution.

Solution:

Part (a) Set \( y(t) = e^{rt} \) and deduce \( r^2 + 2r = 0 \) such that \( r_1 = 0 \) and \( r_2 = -2 \).

This give the homogeneous solution \( y_h(t) = c_1 + c_2 e^{-2t} \)

By the MOUC:

The guess for the particular solutions solving for non homogeneous term \( 4t \) is given as \( y_p(t) = t(\alpha t + \beta) \) where

\[
\begin{align*}
y'_p(t) &= 2\alpha t + \beta \\
y''_p(t) &= 2\alpha 
\end{align*}
\]

Thus \( 2\alpha + 2(2\alpha t + \beta) = 4t \) such that \( \alpha = 1, \beta = -1 \) giving

\[ y_{p1}(t) = t^2 - t \]

The guess for the particular solutions solving for non homogeneous term \( 5 \cos t \) is given as \( y_p(t) = \alpha \cos t + \beta \sin t \) where

\[
\begin{align*}
y'_p(t) &= -\alpha \sin t + \beta \cos t \\
y''_p(t) &= -y_p(t)
\end{align*}
\]

Thus \( -(\alpha \cos t + \beta \sin t) + 2(\beta \cos t - \alpha \sin t) = 5 \cos t \) such that \( 2\beta - \alpha = 5, \beta + 2\alpha = 0 \) giving \( \alpha = -1, \beta = 2 \) and

\[ y_{p2}(t) = -\cos t + 2 \sin t \]

The general solution is given as

\[ y(t) = c_1 + c_2 e^{-2t} + (t^2 - t) + (2 \sin t - \cos t) \]

Part (b) Set \( y(t) = e^{rt} \) and deduce \( r^2 + 2r + 2 = 0 \) such that \( r_1 = \overline{r}_2 = -1 + i \).

This give the homogeneous solution \( y_h(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \)
The non homogeneous term is thus a solution to the homogeneous ODE, by MOUC, we thus have

\[ y_p(t) = t(\alpha e^{-t} \cos t + \beta e^{-t} \sin t) \]

**Problem 2:** (30 points) Suppose a block of mass \( m = 1 \) is attached to a spring with spring constant \( k \). The block slides on a surface with frictional constant \( b = 4 \). Assume the equilibrium position of the block is at \( x = 0 \). Initially, no external forces are present.

(a) (4 points) Write down the equation governing the motion of the block assuming a spring with stiffness constant \( k \) is used.

(b) (4 points) What must be the spring constant \( k \) for the critically damped case?

(c) (14 points) Write down the differential equation for the motion of the block in the critically damped case and solve it to find the position of the block \( x(t) \) as a function of \( t \) using the initial conditions \( x(0) = 2 \) and \( \dot{x}(0) = 1 \). Evaluate the position at \( t = 5 \).

(d) (8 points) Now consider polishing the surface on which the block slides so that we can take the friction constant \( b = 0 \). Consider an external force \( f(t) = \cos(t) \) which acts on the block. We have two springs with \( k = 2 \) and \( k = 1 \) which we can use with our setup. Write down the differential equation for the motion of the block \( x(t) \) in each case. Which spring is more likely to break? Justify your answer.

**Solution:**

(a) The equation of motion is \( ma = mx'' = -bx' - kx \). Substituting in \( m = 1 \) and \( b = 4 \) we obtain:

\[ x'' + 4x' + kx = 0 \]

(b) For critical damping, we must have \( b^2 - 4mk = 0 = 16 - 4k = 0 \implies k = 4 \).

(c) The differential equation, given \( k = 4 \) is then:

\[ x'' + 4x' + 4x = 0 \]

The characteristic equation \( r^2 + 4r + 4 = (r + 2)^2 = 0 \) gives the double root \( r = -2 \). The solution is thus:

\[ x(t) = C_1 e^{-2t} + C_2 te^{-2t} \]

Plugging in the initial conditions we have:

\[ x(0) = C_1 + 0 = 2 \implies C_1 = 2 \]

\[ x'(0) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 te^{-2t} \implies x'(0) = -2C_1 + C_2 = 1 \implies C_2 = 1 + 2C_1 = 5 \]

Thus:

\[ x(t) = 2e^{-2t} + 5te^{-2t} \]

and \( x(5) = 2e^{-10} + 25e^{-10} = 27e^{-10} \). Notice that this is almost zero. This makes sense since a critically damped system quickly returns to its unperturbed state.

(d) The two equations of motion are:

\[ x'' + x = \cos(t) \quad \text{and} \quad x'' + 2x = \cos(t) \]

In the first case \( (k = 1) \), the natural frequency \( \omega_0 = \sqrt{k/m} = 1 \) is the same as the driving frequency of \( f(t) \). Hence, there will be resonant behavior and the displacement is unbounded, that is \( |x(t)| \to \infty \) as \( t \to \infty \) and this spring is likely to break.

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Problem 3: (30 points) Consider the matrix:

\[ A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \]

(a) (7 points) Find all the eigenvalues of matrix \( A \).
(b) (15 points) For each eigenvalue from matrix \( A \), find a basis of the corresponding eigenspace.

\[ B = \begin{bmatrix} 1 & -2 & 3 & -4 \\ 1 & -2 & 3 & -4 \\ 1 & 3 & 3 & 4 \\ 0 & 2 & 3 & 2 \end{bmatrix} \]

(c) (3 points) Find or simply state one of the eigenvalues of matrix \( B \).
(d) (5 points) Consider a 3 \times 3 matrix \( C \) with eigenvalues \( \lambda_1 = 4 \), \( \lambda_2 = -5 \) and \( \lambda_3 = 6 \) with corresponding eigenvectors \( \vec{v}_1 \), \( \vec{v}_2 \) and \( \vec{v}_3 \). Compute \( C\vec{x} \), where \( \vec{x} = 2\vec{v}_1 + \vec{v}_3 \).

Solution:

(a) Matrix \( A \) has a repeated eigenvalue \( \lambda_{1,2} = -1 \) and an eigenvalue \( \lambda_3 = 3 \). To obtain them we compute the determinant of \( A - \lambda I \) by expanding on the first row:

\[
|A - \lambda I| = \begin{vmatrix} -1 - \lambda & 0 & 0 \\ -2 & 1 - \lambda & 2 \\ -2 & 2 & 1 - \lambda \end{vmatrix} = (-1 - \lambda)((1 - \lambda)(1 - \lambda) - 4) \\
= -(1 + \lambda)(\lambda^2 - 2\lambda + 1 - 4) = -(\lambda + 1)(\lambda + 1)(\lambda - 3) = -(\lambda + 1)^2(\lambda - 3)
\]

(b) We compute the eigenvector for \( \lambda_3 \):

\[
(A - \lambda_3 I)\vec{v}_3 = 0 \Rightarrow \begin{bmatrix} -4 & 0 & 0 \\ -2 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\Rightarrow x_1 = 0, \quad x_2 = s, \quad x_3 = s \Rightarrow \vec{v}_3 = s \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Choosing \( s = 1 \), \( \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \) and \( \{\vec{v}_3\} \) forms a basis for the eigenspace of \( \lambda_3 \).

We compute the eigenvectors for \( \lambda_{1,2} \):
\[
(A - \lambda_{1,2} I \vec{v}) = 0 \Rightarrow \begin{pmatrix}
0 & 0 & 0 & 0 \\
-2 & 2 & 2 & 0 \\
-2 & 2 & 2 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
0 & 0 & 0 & 0 \\
-2 & 2 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{cases}
x_1 = r + s \\
x_2 = s \\
x_3 = r
\end{cases} \Rightarrow \vec{v}_1' = \begin{pmatrix} r + s \\ s \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]

Choosing \( s = 1 \) and \( r = 1 \), \( \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) and \( \vec{v}_2 = r \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) and \( \{\vec{v}_1, \vec{v}_2\} \) forms as basis for the eigenspace of \( \lambda_{1,2} \).

(c) The first two rows of matrix \( B \) are identical, thus \( |B| = 0 \) and there is at least one eigenvalue \( \lambda_1 = 0 \). Other eigenvalues would be 1, 6 and -3.

(d) We compute:
\[
C \vec{x} = C(2\vec{v}_1 + \vec{v}_3) = 2C\vec{v}_1 + C\vec{v}_3 = 2\lambda_1 \vec{v}_1 + \lambda_3 \vec{v}_3 = 8\vec{v}_1 + 6\vec{v}_3
\]

Problem 4: (30 points) Please state True or False. No justification necessary.

(a) (6 points) If the general solution of \( y'' + by' + cy = 0 \) is \( y = c_1 e^{\alpha t} + c_2 e^{-\alpha t}, \alpha > 0 \), then one always has \( b = 0 \) and \( c \leq 0 \).

(b) (6 points) The solution to \( y''' + y'' = 0 \) remains bounded as \( t \to \infty \) for any initial condition.

(c) (6 points) A particular solution to \( y'' + 2y' + 2y = e^{-t} \) is \( y = Ate^{-t} \), \( A \in \mathbb{R} \).

(d) (6 points) If the velocity of an unforced damped harmonic oscillator is observed to cross zero at least twice, then the harmonic oscillator cannot be overly damped.

(e) (6 points) If \( \lambda \) is an eigenvalue of \( A \) with eigenvector \( x \), namely \( Ax = \lambda x \), then \( A - 6I \) always has an eigenvalue \( \lambda - 6 \).

Solution:

(a) T.
(b) F.
(c) F.
(d) T.
(e) T.

Problem 5: (30 points) Solve the following problems:

(a) (15 points) Consider the differential equation
\[
t^2 y'' + ty' - 4y = t^2(1 + t^2), \quad t > 0.
\]

(i) Verify that \( y_1(t) = t^2 \) and \( y_2(t) = t^{-2} \) are solutions of the homogeneous equation.

(ii) Calculate the Wronskian of \( y_1 \) and \( y_2 \).

(iii) Using the method of variation of parameters, find a particular solution of the given nonhomogeneous differential equation. Simplify your answer as far as possible.

(iv) Write down the general solution of the given differential equation.
(b) (15 points) Use the method of variation of parameters to find the general solution of
\[ \frac{d^2 y}{dt^2} + y = \sec(t), \quad -\pi/2 < t < \pi/2. \]

**Solution:**
(a) (i) Note that \( y_1'(t) = 2t \) and \( y_1''(t) = 2 \) so that
\[ t^2 y_1'' + ty_1' - 4y_1 = t^2(2) + t(2t) - 4t = 2t^2 + 2t^2 - 4t = 0 \]
and \( y_2'(t) = -2t^{-3}, \) \( y_2''(t) = 6t^{-4} \) so that
\[ t^2 y_2'' + ty_2' - 4y_2 = t^2(6t^{-4}) + t(-2t^{-3}) - 4t^{-2} = 6t^{-2} - 2t^{-2} - 4t^{-2} = 0. \]
Thus \( y_1 \) and \( y_2 \) are solutions of the homogeneous diffeq \( t^2 y'' + ty' - 4y = 0. \)
(ii) \( W[y_1, y_2](t) = \begin{vmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{vmatrix} = t^2(-2t^{-3}) - t^{-2}(2t) = -2t^{-1} - 2t^{-1} = -4t^{-1}. \)
(iii) We seek a solution \( y_p(t) \) of the form \( y_p = v_1 y_1 + v_2 y_2 \) where
\[ v'_1 = -\frac{y_2 f}{W[y_1, y_2]}(t) \quad \text{and} \quad v'_2 = \frac{y_1 f}{W[y_1, y_2]}(t) \]
and \( f(t) = 1 + t^2 \) (note that we have to divide through the given diffeq by \( t^2 \) to identify \( f(t) \): the coefficient on \( y'' \) must be 1, so that \( y'' + (1/t)y' - (4/t^2)y = 1 + t^2 \).
So
\[ v_1(t) = \int \frac{-t^{-2}(1 + t^2)}{-4t^{-1}} dt = \frac{-1}{4} \int t^3(1 + t^2) dt = \frac{-1}{4} \int (t^3 + t^5) dt = \frac{-1}{4} \left( \frac{t^4}{4} + \frac{t^6}{6} \right) = \frac{-1}{16} t^4 - \frac{1}{24} t^6, \]
where we have taken the constant of integration \( C = 0 \) (since we only need *one* function \( v_1 \) that works) and have used the given information \( t > 0 \) to get rid of the absolute value signs in the logarithm. Similarly,
\[ v_2(t) = \int \frac{t^2(1 + t^2)}{-4t^{-1}} dt = \frac{-1}{4} \int t^3(1 + t^2) dt = \frac{-1}{4} \int (t^3 + t^5) dt = \frac{-1}{4} \left( \frac{t^4}{4} + \frac{t^6}{6} \right) = \frac{-1}{16} t^4 - \frac{1}{24} t^6, \]
where once again we have used “\( C = 0 \)”. Finally,
\[ y_p(t) = \left( \frac{1}{4} \ln(t) + \frac{1}{8} t^2 \right) t^2 + \left( -\frac{1}{16} t^4 - \frac{1}{24} t^6 \right) t^{-2} \]
\[ = \frac{t^2}{4} \ln(t) + \frac{t^4}{8} - \frac{t^2}{16} \]
\[ = \frac{t^2}{4} \ln(t) + \frac{t^4}{12} - \frac{t^2}{16} \]
(iv) The general solution is \( y(t) = y_h(t) + y_p(t) \) where \( y_h(t) = c_1 y_1(t) = c_2 y_2(t) \), thus
\[ y(t) = c_1 t^2 + c_2 t^{-2} + \frac{t^2}{4} \ln(t) + \frac{t^4}{12} - \frac{t^2}{16} \]
or
\[ y(t) = c_3 t^2 + c_2 t^{-2} + \frac{t^2}{4} \ln(t) + \frac{t^4}{12}, \]
c_1, c_2, c_3 \in \mathbb{R}, if you combine the two \( t^2 \) terms into one.
(b) The homogeneous diffeq \( y'' + y = 0 \) has linearly independent solutions \( y_1(t) = \cos(t) \) and \( y_2(t) = \sin(t) \) by considering the characteristic equation \( r^2 + 1 = 0 \iff r = \pm i. \)
Note that the Wronskian of $y_1$ and $y_2$ is
\[
W[y_1, y_2](t) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1.
\]
We need a particular solution $y_p(t)$ of the given nonhomogeneous differential equation, which will have the form $y_p = v_1y_1 + v_2y_2$ using variation of parameters, where
\[
v_1' = -\frac{y_2 f}{W[y_1, y_2](t)} \quad \text{and} \quad v_2' = \frac{y_1 f}{W[y_1, y_2](t)} \quad \text{and} \quad f(t) = \sec(t).
\]
Thus
\[
v_1(t) = \int \frac{-\sin(t) \sec(t)}{1} dt = \int \frac{-\sin(t)}{\cos(t)} dt = \ln|\cos(t)| = \ln(\cos(t)),
\]
where we have used the substitution $u = \cos(t) \implies du = -\sin(t)$ and the fact that $\cos(t) > 0$ on the given interval to get rid of the absolute value signs. Similarly,
\[
v_2(t) = \int \frac{\cos(t) \sec(t)}{1} dt = \int 1 dt = t,
\]
where in both integrals we have taken the constant of integration to be $C = 0$.
It follows that a particular solution of the given differential equation is
\[
y_p(t) = \ln(\cos(t)) \cos(t) + t \sin(t),
\]
and with $y_h(t) = c_1y_1(t) + c_2y_2(t)$, $c_1, c_2 \in \mathbb{R}$, the general solution of the differential equation is
\[
y(t) = y_h(t) + y_p(t)
\begin{align*}
&= c_1 \cos(t) + c_2 \sin(t) + \ln(\cos(t)) \cos(t) + t \sin(t) \\
&= (c_1 + \ln(\cos(t))) \cos(t) + (c_2 + t) \sin(t).
\end{align*}