Problem 1: (30 points) Consider the ODE system in the first quadrant \((x \geq 0, y \geq 0)\)

\[
\begin{align*}
\dot{x} &= y - x^2 \\
\dot{y} &= 4(x - y)
\end{align*}
\]

and answer the following questions.

(a) (6 points) Find all equilibrium points.
(b) (10 points) Sketch the horizontal and vertical nullclines and indicate the direction of the vector on these lines.
(c) (8 points) Sketch the direction of the vector field in all regions.
(d) (6 points) Determine the stability of the equilibrium points.

Solution:

(a) The vector field is given by \(V = [y - x^2, 4(x - y)]\). Thus the equilibrium points are determined according to

\[
\begin{align*}
\dot{x} = 0 &\Rightarrow y = x^2 \\
\dot{y} = 0 &\Rightarrow y = x
\end{align*}
\]

Thus \(y = y^2\) and \(y = x\) producing two eq. points \((0, 0), (1, 1)\).

(b) The horizontal nullcline is defined as \(y = x\) such that \(V = [y - y^2, 0]\).

Therefore we find right arrows for \(y < 1\): \(V = [+, 0]\) and left arrows for \(y > 1\): \(V = [-, 0]\).

The vertical nullcline is defined as \(y = x^2\) such that \(V = [0, 4x(1 - x)]\).

Therefore we find up arrows for \(x < 1\): \(V = [0, +]\) and down arrows for \(x > 1\): \(V = [0, -]\).

(c) See figure

(d) Equilibrium point \((0, 0)\) is an unstable source. Equilibrium point \((1, 1)\) is an stable sink.

Problem 2: (30 points) Consider the following linear system:

\[
\begin{align*}
x_1 + x_2 &= \lambda \\
x_3 + x_4 &= 0 \\
3x_1 + x_2 + 2x_3 + 2x_4 &= \lambda \\
3x_1 + x_2 + 2x_3 + \alpha x_4 &= 3
\end{align*}
\]

(a) (2 points) Write down the augmented matrix.

(b) (12 points) Reduce the augmented matrix to upper triangular form.

(c) (6 points) For what value(s) of the real constants \(\alpha\) and \(\lambda\) does the above system have:

(i) Infinitely many solutions?

(ii) No solution?

(iii) One and only one solution?

(d) (4 points) Compute all solutions of the system above for \(\alpha = 0\) and \(\lambda = 1\).

(e) (6 points) Compute all solutions of the system above for \(\alpha = 2\) and \(\lambda = 3\).
Solution:

(a) The augmented matrix for this system is
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & \lambda \\
0 & 0 & 1 & 1 & 0 \\
3 & 1 & 2 & 2 & \lambda \\
3 & 1 & 2 & \alpha & 3
\end{pmatrix}
\]

(b) Doing the following row operations the augmented matrix is reduced to row echelon form (REF)

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & \lambda \\
0 & 0 & 1 & 1 & 0 \\
3 & 1 & 2 & 2 & \lambda \\
3 & 1 & 2 & \alpha & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 & \lambda \\
0 & 0 & 1 & 1 & 0 \\
3 & 1 & 2 & 2 & \lambda \\
0 & 0 & 0 & \alpha - 2 & 3 - \lambda
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 & \lambda \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & -1 & \lambda \\
0 & 0 & 0 & \alpha - 2 & 3 - \lambda
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \lambda \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & \alpha - 2 & 3 - \lambda
\end{pmatrix}
\]

(c) Using the REF:
(i) There are infinitely many solutions if \( \alpha = 2 \) and \( \lambda = 3 \).
(ii) There are no solutions if \( \alpha = 2 \) and \( \lambda \neq 3 \).
(iii) There is a unique solution if \( \alpha \neq 2 \) and any \( \lambda \in \mathbb{R} \).

(d) For \( \alpha = 0 \) and \( \lambda = 1 \) the REF becomes
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & | & 0 \\
0 & 1 & 0 & 0 & | & 1 \\
0 & 0 & 1 & 1 & | & 0 \\
0 & 0 & 0 & -2 & | & 2 \\
\end{pmatrix}
\]
and the RREF becomes
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & | & 0 \\
0 & 1 & 0 & 0 & | & 1 \\
0 & 0 & 1 & 1 & | & 0 \\
0 & 0 & 0 & -2 & | & 2 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & | & 0 \\
0 & 1 & 0 & 0 & | & 1 \\
0 & 0 & 1 & 1 & | & -1 \\
0 & 0 & 0 & 1 & | & -2 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & | & 0 \\
0 & 1 & 0 & 0 & | & 1 \\
0 & 0 & 1 & 0 & | & 1 \\
0 & 0 & 0 & 1 & | & -1 \\
\end{pmatrix}
\]
Thus the unique solution to the system is \( \vec{x} = (x_1, x_2, x_3, x_4) = (0, 1, 1, -1) \).

(e) For \( \alpha = 2 \) and \( \lambda = 3 \) the REF becomes
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & | & 0 \\
0 & 1 & 0 & 0 & | & 3 \\
0 & 0 & 1 & 1 & | & 0 \\
0 & 0 & 0 & 0 & | & 0 \\
\end{pmatrix}
\]
Thus all solutions are given by \( \vec{x} = (x_1, x_2, x_3, x_4) = (0, 3, x_3, -x_3) \) with \( x_3 \in \mathbb{R} \).

**Problem 3: (30 points)**

(a) (12 points) Given the matrices:
\[
A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix},
\]
find the matrix \( C = A^2 + 2AB + B^2 \).

(b) (8 points) For the matrices in part (a), verify that \( (AB)^T = B^T A^T \) by computing both sides.

(c) (10 points) For each of the following sets, state if it is a vector space or not. Justify your answer.

(i) (5 points): The set of vectors \( v \in \mathbb{R}^3 \) such that
\[
\left\{ \begin{array}{l}
v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
\quad \text{with} \quad x^2 + y^2 + z^2 = 1
\end{array} \right\}.
\]

(ii) (5 points): The set of solutions to the differential equation
\[
y' + (\sin^2(t) - t^2) \ y = 0.
\]

**Solution:** (A) We need to evaluate \( A^2 \), \( AB \), and \( B^2 \):
\[
A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix},
\]
\[
AB = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}.
\]
Adding up, we get:
\[ C = A^2 + 2AB + B^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ -5 & 2 \end{bmatrix} \]

Note that this does not equal \((A + B)^2\).

(B) We evaluate:
\[ AB = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} \]

So that:
\[ (AB)^T = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \]

The right hand side is:
\[ B^T A^T = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \]

Hence, \((AB)^T = B^T A^T\).

(C) Here (C1) is not a vector space. Take for instance:
\[ v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

in the space. The vector \(2v\) is not in the space. (C2) is a vector space: it is a homogeneous linear differential equation, all properties of the vector space will hold.
Problem 4: (30 points) Consider the square matrix

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{4} & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{1}{2}
\end{pmatrix}.
\]

(a) (10 points) Compute the determinant of \(A\).

(b) (10 points) Compute the inverse of \(A\) if it exists.

(c) (6 points) Solve \(Ax = b\) for \(x \in \mathbb{R}^3\), where

\[
b = \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}.
\]

You may find it convenient to use \(A^{-1}\).

(d) (4 points) Let \(B\) be the matrix obtained from \(A\) by interchanging the first and third row, namely

\[
B = \begin{pmatrix}
0 & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{4} & 0
\end{pmatrix}.
\]

Is \(B\) invertible? What is the determinant of \(B\)? You do not need to justify your work.

Solution:

(a) The determinant of \(A\) can be computed by expanding for example around the first row:

\[
\det A = (-1)^{1+1} \cdot \frac{1}{2} \cdot \left( \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{4} \right) + (-1)^{1+2} \cdot \frac{1}{4} \cdot \left( \frac{1}{4} \cdot \frac{1}{2} - 0 \cdot \frac{1}{4} \right) = \frac{1}{16}.
\]
(b) The inverse of $A$ can be computed by Gaussian elimination:

\[
[A|I] = \begin{pmatrix}
\frac{1}{2} & \frac{1}{4} & 0 & 1 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1
\end{pmatrix}
\]

\[
R_1 \rightarrow R_1 \\
\rightarrow \begin{pmatrix}
\frac{1}{2} & 0 & 2 & 0 & 0 & 0 \\
\frac{3}{8} & \frac{1}{4} & -\frac{1}{2} & 1 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 1
\end{pmatrix}
\]

\[
R_2 \rightarrow R_2 - \frac{1}{4} R_1 \\
\rightarrow \begin{pmatrix}
1 & 0 & 2 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 2 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 1 & 0
\end{pmatrix}
\]

\[
R_3 \rightarrow R_3 - 3R_2 \\
\rightarrow \begin{pmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \\
-1 & 0 & 1 & -2 & 3
\end{pmatrix}
\]

(c) The solution to $Ax = b$ is

\[
x = A^{-1}b = \begin{pmatrix}
4 \\
-4 \\
4
\end{pmatrix}
\]

(d) The matrix $B$ is still invertible, with $\det B = -\det A = -1/16$.

**Problem 5**: (30 points)

(a) (15 points) Consider the following vectors in $\mathbb{R}^3$:

\[
v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.
\]

The RREF of $A = [v_1 \ v_2 \ v_3]$ is

\[
\begin{pmatrix}
1 & 0 & 7 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{pmatrix}
\]

(i) (6 points) Do the vectors $v_1, v_2, v_3$ form a basis for $\mathbb{R}^3$? Why or why not?

(ii) (5 points) Find all solutions of the equation $Ax = 0$.

(iii) (4 points) Based on your answer in (ii), find the dimension and basis of the subspace consisting of all solutions of the equation $Ax = 0$.

(b) (7 points) Determine whether or not the set of functions $\left\{ \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \begin{bmatrix} 2e^{2t} \\ e^t \end{bmatrix} \right\}$ is linearly independent on all of $\mathbb{R}$.
(c) (8 points) Determine whether or not the set
\[ S = \{ p_1, p_2, p_3, p_4 \} \]
where
\[ p_1(t) = 1, \quad p_2(t) = 1 - t, \quad p_3(t) = 2 - 4t + t^2, \quad p_4(t) = 6 - 18t + 9t^2 - t^3 \]
is a basis for \( \mathbb{P}_3 \).

Solution:
(a) (i) No. The vectors \( v_1, v_2, v_3 \) are not linearly independent. Trying to solve \( c_1v_1 + c_2v_2 + c_3v_3 = 0 \) for real numbers \( c_1, c_2, c_3 \) is the same as solving the equation \( Ax = 0 \), where \( x = (c_1, c_2, c_3) \). Alternatively, it is clear from the RREF that \( \det(A) = 0 \).

(ii) Any solution \( x \) of \( Ax = 0 \), with \( x = (x_1, x_2, x_3) \), must satisfy \( x_1 = -7x_3, \, x_2 = 2x_3, \, x_3 \) free (from the given RREF), so \( x \) must have the form
\[ x = \begin{bmatrix} -7x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}. \]

(iii) The subspace of all solutions to \( Ax = 0 \) is \( \text{Span}\{v\} \) where \( v = (-7, 2, 1) \). It is one dimensional with the single basis vector \( v \).

(b) Yes, they are linearly independent. The vector equation \( c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ e^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) must be true for all \( t \in \mathbb{R} \), so in particular it must be true when \( t = 0 \). Plugging in \( t = 0 \) and rewriting the resulting vector equation in matrix-vector form yields \( \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), which has the unique solution \( \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) since the determinant of the \( 2 \times 2 \) matrix is \(-1 \neq 0\). So the single value \( t = 0 \) is enough to force \( c_1 = c_2 = 0 \).

(c) Yes, it is. Try to solve \( c_1p_1 + c_2p_2 + c_3p_3 + c_4p_4 = 0 \), where here \( 0 \) is the function in \( \mathbb{P}_3 \) given by \( 0(t) = 0 \) for all \( t \in \mathbb{R} \). Plugging in the given functions and equating coefficients yields a system of equations
\[
\begin{align*}
c_1 + c_2 + 2c_3 + 6c_4 &= 0 \\
-c_2 - 4c_3 - 18c_4 &= 0 \\
c_3 + 9c_4 &= 0 \\
-c_4 &= 0
\end{align*}
\]
which obviously has only the trivial solution since the corresponding coefficient matrix is already in upper triangular form. This shows that the four given vectors are linearly independent. But it also shows they span all of \( \mathbb{P}_3 \). Replace the zeros on the right-hand side of the equations above with any real numbers \( a, b, c, d \) and the system is still consistent, so any polynomial \( a + bt + ct^2 + dt^3 \) can be written as a linear combination of the four given polynomials. Hence the given vectors span \( \mathbb{P}_3 \), and together with linear independence this shows that the set \( S \) is indeed a basis for \( \mathbb{P}_3 \).