Problem 1: (30 points) Consider the second order differential equation for a spring-mass system:

\[ m x'' + b x' + k x = 0 \]

with mass \( m \), spring constant \( k \), and damping coefficient \( b \).

(a) Solve the ODE for the initial values

\[ x(0) = x_0, \quad x'(0) = -\frac{b}{2m}x_0, \quad \text{where} \quad b = 2\sqrt{km} \]

and \( x_0 > 0 \).

(b) Classify the type of damping obtained in your solution to (a)

(c) Classify the nature of the damping if \( b = \sqrt{km} \).

(d) What is the natural frequency of the spring? How does this compare to the quasi-frequency in part (c)?

Solution 1:

(a) Characteristic equation \( r_1 = r_2 = -b/2m \) (double root). Therefore

\[ x(t) = (c_1 + c_2t)e^{-\frac{b}{2m}t} \]

\[ x'(t) = -\frac{b}{2m}(c_1 + c_2t)e^{-\frac{b}{2m}t} + c_2e^{-\frac{b}{2m}t} \]

\[ x(0) = (c_1 + c_20) = x_0 \quad \Rightarrow \quad c_1 = x_0 \]

\[ x'(0) = -\frac{b}{2m}(c_1 + c_20) + c_2 = -\frac{b}{2m}x_0 + c_2 = -\frac{b}{2m}x_0 \quad \Rightarrow \quad c_2 = 0 \]

Therefore

\[ x(t) = x_0e^{-\frac{b}{2m}t} \]

(b) Critical damping

(c) Characteristic equation

\[ r_{1,2} = -\frac{b}{2m} \pm i\frac{\sqrt{3}2}{2}\omega_0 \]

The oscillator is now underdamped.

(d) The natural (undamped) frequency of the spring is \( f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \). The quasi-frequency from part (c) is \( f_d = \frac{1}{2\pi}\sqrt{\frac{4mk - b^2}{2m}} = \frac{1}{2\pi}\sqrt{\frac{3k}{4m}}. \)

Problem 2: (30 points) Answer “TRUE” OR “FALSE”. You do NOT need to justify your answer. Only write TRUE if the statement is always true.
(a) Let $A$ be a $2 \times 2$ matrix with repeated eigenvalue $\lambda$. The eigenspace corresponding to $\lambda$ has dimension 1.

(b) Let $A$ be an invertible $n \times n$ matrix with $\lambda$ one of its real eigenvalues. For integer $p > 3$, $\lambda^{-p}$ is an eigenvalue of $A^{-p}$. NOTE: $A^{-p} = A^{-1} A^{-1} \cdots A^{-1}$.

(c) Let $A$ be a $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Then $\mathbb{R}^n = \text{Span} \{ \vec{v}_1, \ldots, \vec{v}_n \}$.

(d) The differential equation $\frac{4}{t} y''(t) + \frac{2}{t} y'(t) - \frac{1}{t} y(t) = (t^2 - 2) \frac{e^t}{t}$ can be solved by the method of undetermined coefficients.

(e) The IVP

\[ y'''(t) + \frac{1}{t - 1} y''(t) + t^3 y'(t) + 3y(t) = 0 \]
\[ y(3) = 1, \quad y'(3) = -1, \quad y''(3) = 0 \]

has a unique solution for $t \in (2, 4)$.

(f) If $u(t)$ and $v(t)$ are solutions to the differential equation $y'' + p(t)y' + q(t)y = f(t)$, where $p(t)$, $q(t)$ and $f(t)$ are continuous functions on interval $I$, then $u(t) + iv(t)$ is also a solution.

Solution 2:

(a) False
(b) True
(c) True
(d) True
(e) True
(f) False

Problem 3: (30 points) Consider the following differential equation

\[ \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 4y = 7e^{-2t}, \]

(a) Find the homogeneous solution.
(b) Use the method of undetermined coefficients to find a particular solution to the differential equation.
(c) What is the general solution?
(d) What is the behavior of the solution as $t \to \infty$?

Solution 3:

(a) The corresponding homogeneous DE is given by

\[ \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 4y = 0. \]

Assuming solutions of the form $e^{rt}$ leads to the characteristic equation

\[ r^2 + 4r + 4 = 0, \]

with a single repeated root given by $r = -2$. The homogeneous solution to the DE is therefore

\[ y_h = c_1 e^{-2t} + c_2 te^{-2t}. \]
(b) Since the forcing function of the DE is itself a solution to the corresponding homogeneous DE, we must guess a particular solution of the form
\[ y_p = At^2e^{-2t} \]

Taking the first and second derivative and substituting into the non-homogeneous DE gives
\[ 2Ae^{-2t} = 7e^{-2t}, \]
which implies \( A = 7/2 \). The particular solution is then
\[ y_p = (7/2)t^2e^{-2t} \]

(c) The general solution is
\[ y = y_h + y_p = c_1e^{-2t} + c_2te^{-2t} + \frac{7}{2}t^2e^{-2t}. \]

(d) In the limit as \( t \to \infty \), \( y(t) \to 0 \), for all values of \( c_1 \) and \( c_2 \).

**Problem 4:** (30 points) Use the differential equation:

\[ (1 - t)y'' + ty' - y = 2(1 - t)^2e^{-t} \]

to answer the following questions.

(a) Which of the following functions is a solution to the homogeneous equation? Show your work and give the homogeneous solution.

(i) \( y_1(t) = t \)
(ii) \( y_2(t) = e^t \)
(iii) \( y_3(t) = e^{-t} \)
(iv) \( y_4(t) = (1 - t)^2 \)

(b) Use your result from part (a) and variation of parameters to find a particular solution to the differential equation.

(c) Find the solution to the initial value problem if \( y(0) = 3/2 \) and \( y'(0) = 1/2 \).

**Solution 4:**

(a) Note that \( y_1 \) and \( y_2 \) satisfy the homogeneous equation \( (1 - t)y'' + ty' - y = 0 \). \( y_3 \) and \( y_4 \) do not. Therefore, the homogeneous solution is given by \( y_h(t) = c_1t + c_2e^t \).

(b) We first note that variation of parameters requires a leading coefficient of 1. Therefore we divide both sides of the differential equation by \( 1 - t \) to obtain

\[ y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = 2(1 - t)e^{-t} \]

Now, let
\[ y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \]
\[ = v_1(t)t + v_2(t)e^t \]

This leads to the system
\[
\begin{bmatrix}
  t & e^t \\
  1 & e^t
\end{bmatrix}
\begin{bmatrix}
  v_1' \\
  v_2'
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  2(1 - t)e^{-t}
\end{bmatrix}
\]
We solve this system using Cramer’s rule and the Wronskian \( W(y_1, y_2) = e^t(t - 1) \) to obtain
\[
v'_1 = \begin{vmatrix}
0 & e^t \\
2(1 - t)e^{-t} & e^t \\
\end{vmatrix}
\frac{2(1 - t)}{e^t(t - 1)} = 2e^{-t}
\]
which implies that \( v_1(t) = -2e^{-t} \). Similarly,
\[
v'_2 = \begin{vmatrix}
t & 1 \\
0 & 2(1 - t)e^{-t} \\
\end{vmatrix}
\frac{2t(1 - t)e^{-t}}{e^t(t - 1)} = -2te^{-2t}
\]
Integrate by parts to obtain \( v_2(t) = te^{-2t} + (1/2)e^{-2t} \).
This leads to:
\[
y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)
= -2te^{-t} + e'(te^{-2t} + (1/2)e^{-2t})
= -te^{-t} + (1/2)e^{-t}
\]
(c) The general solution to the differential equation is now
\[
y(t) = y_h(t) + y_p(t)
= c_1t + c_2e^t - te^{-t} + (1/2)e^{-t}
\]
Since \( y(0) = c_2 + 1/2 = 3/2 \) we obtain \( c_2 = 1 \). Also, \( y'(0) = c_1 + c_2 - 1 - (1/2) = 1/2 \) which gives \( c_1 = 1 \). Therefore, the solution to the IVP is
\[
y(t) = t + e^t - te^{-t} + (1/2)e^{-t}
\]

Problem 5: (30 points) Consider the matrix
\[
A = \begin{pmatrix}
2 & 0 & 1 & 2 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 4 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]
(a) Find all the eigenvalues of \( A \).
(b) For each eigenvalue that you found in part (a), find a basis for the corresponding eigenspace.
(c) What are the eigenvalues of \( A^T \)?

Solution 5:
(a) \( \det(A - \lambda I) = \begin{vmatrix}
2 - \lambda & 0 & 1 & 2 \\
0 & 2 - \lambda & 0 & 0 \\
0 & 0 & 2 - \lambda & 4 \\
0 & 0 & 1 & 2 - \lambda
\end{vmatrix} = (2 - \lambda)(2 - \lambda)[(2 - \lambda)^2 - 4] = 0 \)
\( \Rightarrow (2 - \lambda)^2(4 - \lambda)\lambda = 0 \)
This gives us eigenvalues of \( \lambda_1 = 2 \) (repeated), \( \lambda_2 = 0 \), and \( \lambda_3 = 4 \).
(b) For each eigenvalue we let \( (A - \lambda I)v = 0 \) and then put the resting matrix into RREF.
\[(A - \lambda_1 I) = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \text{RREF} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Since there are two rows of zeros, we have 2 degrees of freedom and expect 2 eigenvectors associated to \(\lambda_1\). The resulting E-vecs are:

\[\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\]

\[(A - \lambda_2 I) = \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \text{RREF} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

The resulting E-vec is:

\[\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}\]

\[(A - \lambda_3 I) = \begin{pmatrix} -2 & 0 & 1 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix} \rightarrow \text{RREF} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

The resulting E-vec is:

\[\vec{v}_4 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}\]

(c) Since the characteristic polynomial for \(A^T\) is identical to that of \(A\), it will have the same eigenvalues: \(\lambda_1 = 2\) (repeated), \(\lambda_2 = 0\), and \(\lambda_3 = 4\).