Problem 1: (35 points) You just bought a cheap car and decide to model its suspension as a mass-spring system.

(a) [7 pts] You measure the car’s mass and its spring and dampening constants and find (after dividing through by the mass) that

\[ x'' + x' + x = 0. \]

Find the general solution for \( x(t) \).

(b) [3 pts] Is your suspension underdamped, critically damped, or overdamped?

(c) [10 pts] You don’t like how your car rides so you adjust the shock absorbers so that

\[ x'' + 2x' + x = 0. \]

Find the general solution for \( x(t) \) and say whether it’s underdamped, critically damped, or overdamped.

(d) [7 pts] You drive up to your friend’s cabin on a washboard dirt road. The corrugated road drives your mass-spring system so

\[ x'' + 2x' + x = 2\sin(t). \]  \hspace{1cm} (1)

Find the general solution for \( x(t) \).

(e) [3 pts] If \( x(0) = 0 \) and \( x'(0) = 1 \), what is the solution of equation (1)?

(f) [5 pts] If your shock absorbers somehow fell off while on the washboard dirt road, so

\[ x'' + x = 2\sin(t), \]

what is the form of the particular solution \( x_p(t) \)?

Solution:

(a) The characteristic equation is \( r^2 + r + 1 = 0 \). Thus, \( \Delta = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0 \). The physical system is critically damped. The solution of the characteristic equation are

\[ r_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \quad r_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i. \]

Therefore, the general solution is

\[ x(t) = e^{-\frac{1}{2}t} \left( c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t \right). \]

(b) Since \( \Delta < 0 \), the system is underdamped.

(c) The characteristic equation is \( r^2 + 2r + 1 = 0 \). Thus, \( \Delta = 2^2 - 4 \cdot 1 \cdot 1 = 0 \). The physical system is critically damped. The solution of the characteristic equation is \( r = -1 \) which is repeated. Therefore, the general solution is

\[ x(t) = c_1 e^{-t} + c_2 te^{-t}. \]

(d) Using the method of undetermined coefficients, we pose \( x_p = A \cos(t) + B \sin(t) \).

Then, plugging \( x_p(t) \) into the DE yields the following equations

\[ (-A + 2B + A) \cos(t) = 0(\cos(t)) \]
\[ (-B - 2A + B) \sin(t) = 2(\sin(t)) \]

We find that \( A = -1 \) and \( B = 0 \). Thus, \( x_p(t) = -\cos(t) \). The general solution is

\[ x_g = x_h + x_p(t) = c_1 e^{-t} + c_2 te^{-t} - \cos(t). \]
(e) From (d)
\[ x(t) = c_1e^{-t} + c_2te^{-t} - \cos(t) \]
\[ x'(t) = -c_1e^{-t} + c_2(e^{-t} - te^{-t}) + \sin(t) \]
So,
\[ x(0) = 0 = c_1 - 1 \]
\[ x'(0) = 1 = -c_1 + c_2 \]
Therefore, \( c_1 = 1, \ c_2 = 2 \). The solution is
\[ x(t) = e^{-t} + 2te^{-t} - \cos(t). \]
(f) Note that \( x(t) = \sin t \) is a homogenous solution. The particular solution has the form
\[ x_p(t) = At \cos t + Bt \sin t. \]

**Problem 2: (20 points)**

(a) [10 pts] Using the method of undetermined coefficients, write down the general form of the particular solution \( y_p(t) \) to
\[ y'' + y' - 6y = f(t) \]
for the following forcing functions:
(i) \( f(t) = \sin 2t + t^2 \)
(ii) \( f(t) = te^{-3t} \)
You do **not** need to solve for the arbitrary coefficients.

(b) [10 pts] Find the general solution of
\[ y'' + y' - 6y = 3e^{2t} \]
using the method of undetermined coefficients.

**Solution:**

(a) (i) The characteristic equation for the homogeneous differential equation
\[ y'' + y' - 6y = 0 \]
is \( r^2 + r - 6 = 0 \). Solving for \( r \), we find \( r = -3 \) and \( r = 2 \). Since the roots are distinct from those of the forcing function, the particular solution has the form
\[ y_p = A \sin 2t + B \cos 2t + Ct^2 + Dt + E. \]
(ii) One of the roots of the forcing function is common to the characteristic equation, so we increase the order of the guess for the particular solution.
\[ y_p = (At^2 + Bt + C)e^{-3t}. \]

(b) The homogeneous equation is the same as in part (a). Thus, the solution to the homogeneous differential equation is
\[ y_h = c_1e^{2t} + c_2e^{-3t}. \]
Since there is a root in common between the forcing function and the characteristic equation, we guess a particular solution of the form \( y_p = (At + B)e^{2t} \). Now, we find the derivatives of this function.

\[
\begin{align*}
y_p &= (At + B)e^{2t} \\
y_p' &= Ae^{2t} + 2(At + B)e^{2t} \\
y_p'' &= 2Ae^{2t} + 2Ae^{2t} + 4(At + B)e^{2t}
\end{align*}
\]

Substitute these into the differential equation, and solve for the coefficients.

\[
2Ae^{2t} + 2Ae^{2t} + 4(At + B)e^{2t} + Ae^{2t} + 2(At + B)e^{2t} - 6(At + B)e^{2t} = 3e^{2t}
\]

\[
2A + 2A + 4(At + B) + A + 2(At + B) - 6(At + B) = 3
\]

\[
4At + 2At - 6At = 0
\]

\[
4A + 4B + A + 2B - 6B = 3
\]

\[
5A = 3
\]

\[
A = \frac{3}{5}
\]

Thus, a particular solution is \( y_p = \frac{3}{5}te^{2t} \). The general solution is

\[
y(t) = y_h + y_p = c_1e^{2t} + c_2e^{-3t} + \frac{3}{5}te^{2t}.
\]

**Problem 3: (25 points) Consider**

\[(1 - t)y'' + ty' - y = (t - 1)^2e^t.\]

(a) [5 pts] Verify that \( y_1(t) = t \) and \( y_2(t) = e^t \) are both solutions to the corresponding homogeneous differential equation.

(b) [15 pts] Find a particular solution using the variation of parameters method and determine the general solution.

(c) [5 pts] If \( y(0) = 1 \) and \( y'(0) = 2 \), determine the constants in the general solution.

**Solution:**

(a) For \( y_1(t) = t \),

\[
y_1'(t) = 1, \quad y_1''(t) = 0
\]

\[
(1 - t)y_1'' + ty_1' - y_1 = t - t = 0
\]

Thus, \( y_1(t) \) solves the homogeneous differential equation.

For \( y_2(t) = e^t \),

\[
y_2'(t) = y_2''(t) = e^t
\]

\[
(1 - t)y_2'' + ty_2' - y_2 = (1 - t)e^t + te^t - e^t = 0
\]

Thus, \( y_2(t) \) solves the homogeneous differential equation.

(b) Assume a particular solution

\[
y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t).
\]

The Wronskian

\[
W(y_1, y_2) = y_1y_2' - y_1'y_2 = te^t - e^t.
\]
The forcing $f(t) = \frac{(t-1)^2 e^t}{1-t} = -(t - 1)e^t$. Thus,

\[ v'_1(t) = \frac{-y_2 f}{W(y_1, y_2)} = \frac{e^{2t}(t-1)}{te^t - e^t} = e^t, \]

\[ v'_2(t) = \frac{y_1 f}{W(y_1, y_2)} = \frac{-t(t-1)e^t}{te^t - e^t} = -t, \]

and $v_1(t) = e^t, v_2(t) = -\frac{1}{2} t^2$. So a particular solution is

\[ y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = te^t - \frac{1}{2} t^2 e^t. \]

The general solution is

\[ y_g(t) = y_h + y_p = c_1 t + c_2 e^t + te^t - \frac{1}{2} t^2 e^t, \]

for arbitrary constants $c_1, c_2$.

(c) Plugging 0 for $t$ in $y_g$,

\[ c_2 e^0 = 1, \quad c_2 = 1. \]

And

\[ y'_g(t) = c_1 + c_2 e^t + e^t + te^t - t e^t - \frac{1}{2} t^2 e^t = c_1 + 2e^t - \frac{1}{2} t^2 e^t. \]

Thus, $y'(0) = 2$ implies

\[ c_1 + 2 = 2, \quad c_1 = 0. \]

Therefore, $c_1 = 0, c_2 = 1$.

**Problem 4:** (20 points)

(a) [5 pts] Let $A$ be a $2 \times 2$ matrix with eigenvalue/eigenvector pairs $\lambda_1 = 2, \vec{v}^{(1)} = (1, 2)^T$ and $\lambda_2 = -1, \vec{v}^{(2)} = (3, 4)^T$. Also let $\vec{x} = \vec{v}^{(1)} + 2\vec{v}^{(2)}$. What is $A^3 \vec{x}$?

(b) [5 pts] Consider a matrix

\[ B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} \]

which has an eigenvalue/eigenvector pair of $\lambda_1 = 10$ and $\vec{v}^{(1)} = (1, 2, 3, 4)^T$. Find all other eigenvalues. Be sure to justify your reasoning. HINT: You shouldn’t have to actually compute the eigenvalues and eigenvectors. This question can be answered simply by knowing the properties of matrices.

(c) [10 pts] For the matrix

\[ C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

find the basis vectors for all the eigenspaces.

**Solution:**
(a) Since we have been given the eigenvalue/eigenvector pairs, we can compute the following
\[
A^3 \vec{x} = A^3 (\vec{v}^{(1)} + 2 \vec{v}^{(2)}) \\
= A^2 (A \vec{v}^{(1)} + 2 A \vec{v}^{(2)}) \\
= A^2 (2 \vec{v}^{(1)} - 2 \vec{v}^{(2)}) \\
= A (4 \vec{v}^{(1)} + 2 \vec{v}^{(2)}) \\
= 8 \vec{v}^{(1)} - 2 \vec{v}^{(2)} \\
= 8 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.
\]

(b) We know that another eigenvalue is zero because the matrix is singular. The RREF for \( \lambda_2 = 0 \) yields a 3-dimensional eigenspace, so \( \lambda_1 = 10 \) and \( \lambda_2 = 0 \) are the only two eigenvalues.

(c) The eigenvalues are \( \lambda_1 = 0 \) and \( \lambda_2 = 2 \). The corresponding eigenspaces are

\[
E_{\lambda_1} = \left\{ s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, s, r \in \mathbb{R} \right\}
\]

with basis

\[
\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

and

\[
E_{\lambda_2} = \left\{ s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, s \in \mathbb{R} \right\}
\]

with basis

\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.
\]