Problem 1: (15 points) Consider an unforced mass-spring system with mass $m = 2$, spring constant $k = 8$, and damping constant $b = 4p$, where $p > 0$ is a constant.

(a) [4] Write down the differential equation that models this physical system.

(b) [5] For different values of $p$, the solutions will behave differently. Find the threshold value for $p$ that separates one solution behavior from a different solution behavior. NOTE: one qualitative change occurs when $p = 0$, but we assume $p > 0$ in this problem.

(c) [6] Sketch the solution for the mass-spring system when $p = 1$. Use the initial conditions $x(0) = 2$, $x'(0) = 0$.

Solution:

(a) $2x'' + 4px' + 8x = 0$

(b) The solution behavior will change when the system becomes critically damped (i.e., the system no longer oscillates). This will happen when the discriminant is equal to zero: $(4p)^2 - 4(2)(8) = 0 \Rightarrow p = 2$.

(c) Setting $p = 1$, we solve the differential equation. The characteristic equation yields: $r = -1 \pm i\sqrt{3}$. The solution is

$$x(t) = e^{-t}(c_1 \cos(t\sqrt{3}) + c_2 \sin(t\sqrt{3})).$$

Applying the initial conditions gives: $c_1 = 2$, $c_2 = \frac{2}{\sqrt{3}}$.

Note that the phase portrait looks like
Problem 2: (20 points) Consider the following differential equation.

\[ ty'' + (t - 1)y' - y = t^2, \quad t > 0 \]  

(a) [5] Verify that \( y_1(t) = 1 - t \) and \( y_2(t) = e^{-t} \) are both solutions to the related homogeneous differential equation.

(b) [15] Find the general solution, \( y(t) \), of equation (1).

Solution:

(a) Check that each function satisfies the homogeneous equation.

\( y_1: \)

\[ ty_1'' + (t - 1)y_1' - y_1 = t(1 - t)'' + (t - 1)(1 - t)' - (1 - t) \]
\[ = t(0) + (t - 1)(-1) - (1 - t) \]
\[ = 0 \]

\( y_2: \)

\[ ty_2'' + (t - 1)y_2' - y_2 = t(e^{-t})'' + (t - 1)(e^{-t})' - e^{-t} \]
\[ = te^{-t} + (t - 1)(-e^{-t}) - e^{-t} \]
\[ = 0 \]

(b) Solve by variation of parameters. First, we must get into the correct form by dividing through by \( t \):

\[ y' + \frac{t - 1}{t}y' - \frac{1}{t^2}y = t \]

Next, compute the Wronskian of \( y_1 \) and \( y_2 \):

\[ W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 - t & e^{-t} \\ -1 & -e^{-t} \end{vmatrix} = te^{-t} \]
Now, using the formulas for $v'_1$ and $v'_2$:

$$v'_1 = -\frac{y_2 f}{W[y_1, y_2]} = -\frac{e^{-t}t}{te^{-t}} = -1$$

So $v_1 = -t$.

$$v'_2 = \frac{y_1 f}{W[y_1, y_2]} = \frac{(1-t)t}{te^{-t}} = (1-t)e^t$$

We integrate to find $v_2$:

$$v_2 = \int (1-t)e^t \, dt$$

$$= e^t - (te^t - e^t)$$

$$= 2e^t - te^t$$

The general solution is given by

$$y(t) = y_h(t) + y_p(t)$$

$$= c_1 y_1(t) + c_2 y_2(t) + v_1(t)y_1(t) + v_2(t)y_2(t)$$

$$= c_1(1 - t) + c_2 e^{-t} - t(1 - t) + (2e^t - te^t)e^{-t}$$

$$= c_1(1 - t) + c_2 e^{-t} + t^2 - 2t + 2$$

(c) The solution can also be found using the method of undetermined coefficients since both the forcing function and the variable coefficients are polynomials. If we pose $y_p = At^2 + Bt + C$, then

$$2At + (t - 1)(2At + B) - (At^2 + Bt + C) = t^2$$

$$\Rightarrow A = 1 \text{ and } B = -C$$

We can choose $C = 0$, and so $B = 0$, which leads to $y_p(t) = t^2$. Note: we can not uniquely determine C precisely because $1 - t$ is a solution of the associated homogeneous equation.

Problem 3: (20 points) Consider the linear differential equation

$$y'' + 4y' + 5y = f(t). \tag{2}$$

Using the Method of Undetermined Coefficients, write down the general form of the particular solution, $y_p$, for the following forcing functions. **You do NOT** need to solve for the arbitrary coefficients.

(a) $f(t) = t + e^{-2t}$
(b) \( f(t) = e^t \cos(-2t) \)
(c) \( f(t) = e^{-2t} \sin(t) \)
(d) \( f(t) = te^{-2t} \cos(t) \)

**Solution:**

First, we find the homogenous solution, \( y_h \). The characteristic equation yields:

\[ r^2 + 4r + 5 = 0 \Rightarrow r = -2 \pm i. \]

Thus, \( y_h(t) = e^{-2t}(c_1 \cos(t) + c_2 \sin(t)) \). For the given \( f(t) \), the form of the particular solution, \( y_p \), is:

(a) \( y_p = (At + B) + Ce^{-2t} \)
(b) \( y_p = e^t(A \cos(-2t) + B \sin(-2t)) \)
(c) \( y_p = te^{-2t}(A \cos(t) + B \sin(t)) \)
(d) \( y_p = te^{-2t}((At + B) \cos(t) + (Ct + D) \sin(t)) \)

**Problem 4:** (20 points)

(a) [8] Find the eigenspace(s) for the eigenvalues of the following matrix:

\[
A = \begin{bmatrix}
0 & 1 & -1 \\
0 & -1 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

**Solution:**

(i) **Eigenvalues:** The characteristic equation is

\[ |A - \lambda I| = -\lambda^2(1 + \lambda) = 0 \]

which means that \( \lambda = 0, -1 \) with 0 a repeated root.

(ii) **Eigenspaces:**

(i) For \( \lambda = 0 \), calculate the RREF of \([A - 0I][\vec{0}]\) to obtain

\[
\begin{bmatrix}
0 & 1 & -1 & | & 0 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

which yields two eigenvectors

\[ \vec{v}^{(1)} = (1, 0, 0)^T \]

and

\[ \vec{v}^{(2)} = (0, 1, 1)^T. \]

Thus the eigenspace is

\[ E_{\lambda=0} = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; \alpha, \beta \in \mathbb{R} \right\} \]

(ii) For \( \lambda = -1 \), calculate the RREF of \([A - (-1)I][\vec{0}]\) to obtain

\[
\begin{bmatrix}
1 & 1 & 0 & | & 0 \\
0 & 0 & 1 & | & 0 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]
which yields one eigenvector
\[ \vec{v}^{(3)} = (1, -1, 0)^T. \]

Thus, the eigenspace is
\[ E_{\lambda=-1} = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}. \]

(b) [8] Find all eigenvalues for the following matrix
\[ A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} \]

HINT: i) One of the eigenvectors is \( \vec{v} = (1, 1, 1, 1, 1)^T \) and ii) you shouldn’t have to solve \( |A - \lambda I| = 0 \) to identify the eigenvalues.

Solution: The eigenvalue for \( \vec{v} \) is 10 and the other eigenvalue is zero. The RREF for \( \lambda = 0 \) yields a 4-dimensional eigenspace, so \( \lambda = 0 \) and \( \lambda = 10 \) are the only two eigenvalues.

(c) [4] Find the general form of all \( 2 \times 2 \) matrices for which \( (1, 1)^T \) is an eigenvector.

Solution: We know that
\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
which yields the two equations
\[ a + b = \lambda \]
\[ c + d = \lambda \]

So, all \( 2 \times 2 \) matrices that satisfy this requirement will look like:
\[ \begin{bmatrix} a & b \\ a + b - d & d \end{bmatrix}. \]

Carried a bit further, we observe that \( a + d = 2\lambda \), since the trace of a matrix equals the sum of its eigenvalues and this matrix has a repeated eigenvalue. Using this, and adding together \( a + b = \lambda \) and \( c + d = \lambda \), we see that \( b + c = 0 \). Since \( c = -b \) and \( a + b = c + d \), we have that \( 2b = d - a \) or \( b = \frac{d-a}{2} \). Thus, the matrix can also be written:
\[ \begin{bmatrix} a & \frac{d-a}{2} \\ -\frac{d-a}{2} & d \end{bmatrix}. \]

Problem 5: (25 points) Give a brief answer to each question. Box your answers. Each correct answer earns 5 points. No work for this question will be graded.

(a) Let \( A \) be a \( 2 \times 2 \) matrix. The eigenvalues of \( A \) are \( \lambda = 3 \) and \( \lambda = 2 \). What are the eigenvalues of \( AA^{-1} \)?

(b) Let \( x(t) \) be a function of \( t \), and let \( x'' - x = 0 \) with \( x(0) = 1, x'(0) = 0 \). Describe the long-term behavior of \( x(t) \).
(c) Find a basis for the solution space of \( x'' - x = 0 \).

(d) Consider \( x'' - x = 0 \). Set \( y_1 = x \) and \( y_2 = x' \). Find the matrix \( A \) so that
\[
\begin{bmatrix}
y'_1 \\
y'_2
\end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

(e) (TRUE/FALSE) If \( \vec{u} \) is an eigenvector of \( A \) and \( \vec{v} \) is an eigenvector of \( B \), then \( \vec{u} + \vec{v} \) is an eigenvector of \( A + B \).

**Solution:**

(a) The eigenvalues of \( AA^{-1} \), \( \lambda \), are given by the solutions to \( |AA^{-1} - \lambda I| = 0 \). Since \( AA^{-1} = I \) and \( |I| = 1 \),
\[
|AA^{-1} - \lambda I| = \\
|I - \lambda I| = \\
|I(1 - \lambda)| = \\
|I(1 - \lambda)| = 0
\]

This holds only when \( \lambda = 1 \).

Alternatively, note that \( AA^{-1} = I \). Since the eigenvalues of a diagonal matrix lie on the main diagonal, we see that the eigenvalues of \( AA^{-1} \) are all equal to one.

(b) The characteristic equation of \( x'' - x = 0 \) is \( r^2 - 1 = 0 \). Thus, \( x(t) = c_1 e^t + c_2 e^{-t} \). The initial conditions yield \( c_1 = c_2 = 1/2 \). The long-term behavior is that \( x(t) \to \infty \).

(c) A basis for the solution space is \( \{e^t, e^{-t}\} \).

(d) Setting \( y_1 = x \), \( y_2 = x' \), we see from the differential equation that \( y'_2 = y_1 \). Using the fact that \( y'_1 = y_2 \), we arrive at the system
\[
\begin{bmatrix}
y'_1 \\
y'_2
\end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

(e) FALSE. Let \( A\vec{u} = \lambda_1 \vec{u} \) and \( B\vec{v} = \lambda_2 \vec{v} \). Then, \( (A+B)(\vec{u}+\vec{v}) = (A+B)\vec{u} + (A+B)\vec{v} = \lambda_1 \vec{u} + \lambda_2 \vec{v} + B\vec{u} + A\vec{v} \) which is not necessarily equal to \( \lambda_3 (\vec{u} + \vec{v}) \).