ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor’s name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and box in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, calculators and electronic devices of ANY sort are NOT permitted. One 8" × 11", two-sided, sheet of notes is allowed.

1. (8 points) Let \( \vec{u}_1, \vec{u}_2 \) and \( \vec{u}_3 \) be the vectors in \( \mathbb{R}^3 \) given by

\[
\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 3 \\ k \end{bmatrix},
\]

where \( k \) is a constant. Find all the values of \( k \) for which \( \vec{u}_1, \vec{u}_2 \) and \( \vec{u}_3 \) are linearly independent.

**Solution**

The vectors are linearly independent if and only if the determinant of the 3 × 3 matrix \( A \) whose columns are \( \vec{u}_1, \vec{u}_2, \vec{u}_3 \) is not zero. This determinant is, evaluating it using the left column,

\[
\begin{vmatrix}
1 & 1 & 0 \\
0 & 2 & 3 \\
1 & 2 & k
\end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 2 & k \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 1(2k - 6) + 1(3 - 0) = 2k - 3. \tag{1}
\]

So the vectors are linearly independent when \( k \neq 3/2 \).

2. (18 points) Consider the following system of equations for the variables \( x_1, x_2, x_3, x_4 \):

\[
\begin{align*}
x_1 + x_2 + x_3 &= 3, \\
2x_1 - x_2 - x_3 + 6x_4 &= 0, \\
x_1 - 2x_2 - 2x_3 + 6x_4 &= -3.
\end{align*}
\]

(a) (9 points) Write the system in augmented matrix form and transform it to Reduced Row Echelon Form (RREF) using row operations.

**Solution**

The augmented matrix for this system, and operations that reduce it to RREF, are

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & | & 3 \\
2 & -1 & -1 & 6 & | & 0 \\
1 & -2 & -2 & 6 & | & -3
\end{bmatrix} \rightarrow R_3 - R_1 \rightarrow R_3 \rightarrow R_2 - 2R_1 \rightarrow R_2 \tag{2}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & | & 3 \\
0 & -3 & -3 & 6 & | & -6 \\
0 & -3 & -3 & 6 & | & -6
\end{bmatrix} \rightarrow R_3 - R_2 \rightarrow R_3 \rightarrow R_2 / (-3) \rightarrow R_2 \tag{3}
\]
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  \quad \text{(RREF)}  \quad (4)

(b) (9 points) Find all solutions of the system or prove that no solutions exist.

\textbf{Solution} There is a consistent zero row, two pivot variables, and four variables, so the system has infinite solutions. Letting \( x_3 = r \), \( x_4 = s \), the solutions can be written as
\[
x_1 = 1 - 2s, \quad x_2 = 2 - r + 2s, \quad x_3 = r, \quad x_4 = s.
\]
In vector form,
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
1 \\
2 \\
0 \\
0
\end{bmatrix} + r \begin{bmatrix}
0 \\
-1 \\
1 \\
0
\end{bmatrix} + s \begin{bmatrix}
-2 \\
2 \\
0 \\
1
\end{bmatrix}.
\]

3. (16 points) A competition model between two species is given by
\[
\begin{align*}
u' &= u(4 - u - 2v), \\
v' &= v(4 - 2u - v),
\end{align*}
\]
where \( u \) and \( v \) represent the population of the two species.

(a) Determine any equilibria for this system of equations.

(b) Sketch a phase-plane diagram for this system, including all of the nullclines, in an appropriate region of phase space. Include the direction arrows on the nullclines. Sketch some typical solutions.

\textbf{Solution}

(a) The equilibria \((u, v)\) are the solutions of
\[
\begin{align*}
0 &= u(4 - u - 2v), \quad \text{(5)} \\
0 &= v(4 - 2u - v), \quad \text{(6)}
\end{align*}
\]
If \( v = 0 \), we must have \( u = 0 \) or \( u = 4 \), so that gives \((0, 0)\) and \((4, 0)\). Similarly, if \( u = 0 \) then \( v = 0 \) or \( v = 4 \), which gives the additional equilibrium \((0, 4)\). The other possibility is that \( 0 = 4 - u - 2v \) and \( 0 = 4 - 2u - v \), which gives \((4/3, 4/3)\).

(b) The relevant region in phase space is \( u \geq 0 \), \( v \geq 0 \) since \( u \) and \( v \) represent populations and are therefore nonnegative. The nullclines and the phase diagram are sketched below.
Nullclines:
\[ U' = 0 \Rightarrow V = 0 \text{ or } V = 2 - \frac{4}{2} \]
\[ V' = 0 \Rightarrow V = 0 \text{ or } V = 4 - 2U \]
4. (24 points) TRUE/FALSE questions. No explanations are needed. Make sure to write down whole words “TRUE” or “FALSE”. Only write “TRUE” if the statement is always true. Each question is worth 4 points.

(a) If $A$, $B$, $C$ are $n \times n$ matrices and $|ABC| = 0$, then at least one of the matrices $A$, $B$, $C$ must be noninvertible.

(b) The sum of two $n \times n$ invertible matrices is invertible.

(c) If a $n \times n$ matrix $A$ is noninvertible, the system $A\vec{x} = \vec{b}$ has no solutions.

(d) The polynomials $p_1(t) = t + 1$ and $p_2(t) = t - 1$ form a basis for the vector space of all polynomials of degree 1 or less, $P_1 = \{a + bt \text{ with } a, b \in \mathbb{R}\}$.

(e) The functions $\sin(t)$ and $\sin(2t)$ are linearly independent on the interval $[-1, 1]$. You might find useful that $\sin(2t) = 2 \sin(t) \cos(t)$ and $\cos(2t) = 1 - 2 \sin^2(t)$.

(f) $|kI| = k^4$ for $k$ a real number and $I$ a $4 \times 4$ identity matrix.

Solutions

(a) TRUE.

(b) FALSE. (Consider $I$ and $-I$.)

(c) FALSE. (Could have infinite solutions.)

(d) TRUE.

(e) TRUE.

(f) FALSE. ($|kI| = k^4.$)

5. (22 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Evaluate each expression or explain why it is not defined.

(a) (2 points) $AB$.

(b) (2 points) $BA$.

(c) (2 points) $|B + 2I|$.

(d) (2 points) $A^T$.

(e) (2 points) $A + B$.

(f) (4 points) All $\vec{x}$ in $\mathbb{R}^2$ such that $B\vec{x} = 0$.

(g) (4 points) The span of the columns of $A$.

(h) (4 points) $(kB)^{-1}$ where $k > 0$.

Solution

(a) $AB = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$. 
(b) $BA$ is not defined since $B$ is $2 \times 2$ and $A$ is $3 \times 2$.

(c) $|B + 2I| = 7$. ($|A + B| \neq |A| + |B|$ !!)

(d) $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(e) $A + B$ is not defined since $B$ is $2 \times 2$ and $A$ is $3 \times 2$.

(f) $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $B$ is invertible.

(g) Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. The columns of $A$ are linearly independent, and therefore they span a plane in $\mathbb{R}^3$, but this plane is not $\mathbb{R}^2$.

(h) $\frac{1}{k} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

6. (12 points) Consider the vector space $V = \{2 \times 2$ matrices with real-valued elements$\}$. Prove that the following are or are not vector subspaces of $V$.

(a) (4 points) $W = \{A$ in $V$ such that All diagonal elements of $A$ are zero$\}$.

(b) (4 points) $W = \{A$ in $V$ such that $A^T = -A$\}.

(c) (4 points) $W = \{A$ in $V$ such that Trace($A$) = $|A|$\}.

[Solution] In this problem, to prove that $W$ is a vector subspace, we have to show for two arbitrary matrices $A \in W$ and $B \in W$ the matrix $\alpha A + \beta B \in W$ for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. To prove that $W$ is not a vector subspace, we just need to find an example with $\alpha$, $\beta$, $A$, and $B$ such that $\alpha A + \beta B \notin W$.

In all cases, $\alpha A + \beta B$ results in a $2 \times 2$ matrix, and it is thus in $V$. Therefore, we know that (for each of the parts of the problem) $W$ is a subspace of $V$. However, it remains to be proven as to whether or not $W$ is a vector subspace of $V$.

(a) Consider the following matrices which are elements of $W$:

\[
A = \begin{bmatrix} 0 & a_{21} \\ a_{12} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b_{21} \\ b_{12} & 0 \end{bmatrix}
\]

We note that the matrix

\[
\alpha \begin{bmatrix} 0 & a_{21} \\ a_{12} & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & b_{21} \\ b_{12} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a_{21} + \beta b_{21} \\ \alpha a_{12} + \beta b_{12} & 0 \end{bmatrix}
\]

has zeros on the diagonal for all values of $\alpha$ and $\beta$. Thus we can conclude that $W$ is a vector subspace of $V$. 


(b) Consider the matrices $A$ and $B$ as elements of $W$. Therefore, we know that $A^T = -A$ and $B^T = -B$. If we consider

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$$

$$= -\alpha A - \beta B$$

$$= -(\alpha A + \beta B)$$

then we have shown that the matrix $\alpha A + \beta B$ (for all values of $\alpha$ and $\beta$) is also in $V$. Thus we can conclude that $W$ is a vector subspace of $V$.

(c) In this case, if we consider the two matrices $A$ and $B$ again, we quickly get bogged down in trying to prove that $\text{Trace}(\alpha A + \beta B) = |\alpha A + \beta B|$. So, we search for an example which will disprove this statement. Here is one:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\alpha = 2$, $\beta = 0$. We see that $\text{Trace}(2A) = 2$ while $|2A| = 4$. Therefore $W$ is not a vector subspace.