Problem 1: (20 points) Consider the following model for the competition between Buffalo (B) and Bighorn Rams (R).

\[ B' = -B^2 + BR + B \]
\[ R' = 2R - BR \]

(a) (8 points) Compute all the equilibrium points for this system.
(b) (10 points) Copy the following figure into your exam book.

On your figure, label the equilibrium points. The two non-axis lines represent two of the four nullclines. Find the equations for all of the nullclines, label them on your figure, and draw arrows to indicate in which direction the flow moves across the nullclines. Guesses will not receive partial credit.

(c) (2 points) Justify why or why not these two species can coexist (according to this model).
Solution:

(a) Equilibrium points are \((B, R) = (0, 0), (1, 0), \) and \((2, 1)\).
(b) The figure below depicts the full phase plane. The dotted lines are the \(R' = 0\) nullclines and the solid lines are the \(B' = 0\) nullclines.
(c) Yes, they can coexist, because there is an equilibrium point at \((2,1)\).

Problem 2: (20 points) Consider the following matrix and vector,

\[
A = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
2 & 4 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 \\
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}
2 \\
3 \\
\lambda \\
\end{bmatrix}.
\]

(a) (10 points) \(\text{Col}(A)\) is the span of the column vectors of matrix \(A\). For what values of \(\lambda\) is \(\vec{b} \in \text{Col}(A)\).
(b) (2 points) Find the dimension of \(\text{Col}(A)\).
(c) (5 points) Find the general solution to the problem \(A\vec{x} = \vec{0}\).
(d) (3 points) Define the set \(\text{Null}(A)\), or \(\text{N}(A)\), as

\[
\text{N}(A) = \{ \vec{x} \in \mathbb{R}^5 : A\vec{x} = \vec{0} \}.
\]

Construct a basis for \(\text{N}(A)\) and find the dimension of \(\text{N}(A)\).

Solution:

(a)

\[
[A | \vec{b}] = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 & 2 \\
2 & 4 & 1 & 1 & 1 & 3 \\
0 & 0 & 2 & 2 & 2 & \lambda \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 1 & 1 & -1 \\
0 & 0 & 2 & 2 & 2 & \lambda \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & \lambda + 2 \\
\end{bmatrix}
\]

Therefore \(\vec{b} \in \text{Col}(A)\) is equivalent to that \(A\vec{x} = \vec{b}\) is consistent.

Thus \(\lambda = -2\).

(b) \(\text{dim}(\text{Col}(A)) = \text{Rank}(A) = \# \text{ pivots} = 2\).

(c) \(x_2, x_4, x_5\) are free variables. Let \(x_2 = l, x_4 = s, x_5 = t\).

\[
\vec{x} = \begin{bmatrix}
-2l \\
l \\
-s - t \\
\end{bmatrix} = l \begin{bmatrix}
-2 \\
1 \\
0 \\
\end{bmatrix} + s \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix} + t \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}, \text{ for any } l, s, t \in \mathbb{R}.
\]
(d) \( \dim(\text{Null } A) = \# \text{ free variables} = 3. \)

A basis is the set
\[
\begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
-1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
-1 \\
1
\end{bmatrix}
\]

Problem 3: (20 points)

(a) (5 points) Let 
\[
A = \begin{pmatrix}
2 & 7 & 8 \\
0 & 3 & 9 \\
0 & 0 & 1
\end{pmatrix}, \quad
D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Find \( \det(A^T D^{-1} A) \).

(b) (5 points) Are the vectors 
\[
\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad
\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad
\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad
\vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

linearly independent? Show all work. If said vectors are linearly independent, what is their span?

(c) (5 points) Are the polynomials \( t^2 - 2t + 1, t + 1, \) and \( 3t + 2 \) linearly independent? Show all work.

(d) (5 points) Assume matrices A, B, and C are invertible. Simplify the expression
\[
A^{-2} (B^2 C^{-1} A^{-2})^{-1} B^2
\]
so that only one term remains.

Solution:

(a) \( \det(A^T D^{-1} A) = \det(A)^2 / \det(D) = 18. \)

(b) Form matrix with columns as the \( \vec{v}_j \)'s. Determinant is -3. Thus four vectors are linearly independent, and thus they form a basis for \( \mathbb{R}^4. \)

(c) The Wronskian is
\[
W = \begin{vmatrix}
t^2 - 2t + 1 & t + 1 & 3t + 2 \\
2t - 2 & 1 & 3 \\
2 & 0 & 0
\end{vmatrix} = 2,
\]

so the answer is yes, they are linearly independent.

(d) \( A^{-2} (B^2 C^{-1} A^{-2})^{-1} B^2 = A^{-2} A^2 C B^{-2} B^2 = C. \)

Problem 4: (20 points) Let \( M_{22} \) denote the space of all \( 2 \times 2 \) matrices. Let \( V \in M_{22} \) be some unspecified, invertible matrix. Define the subset of \( M_{22} \)
\[
W = \{ A \in M_{22} : A = V^{-1} DV \} \]
where $D$ can be any diagonal $2 \times 2$ matrix. For instance $V^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}V$ and $V^{-1}\begin{pmatrix} 23 & 0 \\ 0 & 4.9 \end{pmatrix}V$ are elements of $W$.

(a) (10 points) Is $W$ a subspace of $M_{22}$? Justify your answer.
(b) (5 points) Prove that $W$ is closed under matrix multiplication i.e. if $A,B \in W$, then $AB \in W$.

$\text{Hint: Show that the product of two } (2 \times 2) \text{ diagonal matrices is also diagonal.}$
(c) (5 points) Show that $AB = BA$.

Solution:

(a) Let $A = V^{-1}D_1V$ and $B = V^{-1}D_2V$. Then $A + B = V^{-1}(D_1 + D_2)V$, the sum of two diagonal matrices is diagonal, and therefore the set is closed under addition. Likewise, $cA = V^{-1}(cD_1)V$, $cD_1$ is diagonal, and therefore $W(V)$ is closed under scalar multiplication. Thus $W(V)$ is a subspace of $M_{22}$.

(b) Again, let $A = V^{-1}D_1V$ and $B = V^{-1}D_2V$. Then $AB = V^{-1}D_1D_2V$.

Let $D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{12} \end{pmatrix}$ and $D_2 = \begin{pmatrix} d_{21} & 0 \\ 0 & d_{22} \end{pmatrix}$. Thus $D_1D_2 = \begin{pmatrix} d_{11}d_{21} & 0 \\ 0 & d_{12}d_{22} \end{pmatrix}$, which is diagonal, and so we are done.

(c) From above, $D_1D_2 = D_2D_1$, so then $AB = BA$.

Problem 5: (20 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ k & 2 & 3 \\ 3 & 4 & k \end{bmatrix}$$

(a) (5 points) For what values of $k$ does $A^{-1}$ exist?
(b) (5 points) If $k$ equals one of the problematic values for which $A^{-1}$ does not exist, what is the rank of $A$? (Don’t forget to justify your answer.)
(c) (10 points) Assuming that $k = 2$, compute $A^{-1}$.

Solution:

(a) The determinant of $A$ is $-(k - 3)^2$ and thus the problematic value is $k = 3$.

(b) For $k = 3$,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix}$$
and the rank is 2 since the first and second columns are linearly independent.

(c) To find the inverse, we must solve the following system

\[ AX = I \]

where \( I \) is the 3 \times 3 identity matrix and \( X \) will be the inverse of \( A \). So, we need to row-reduce to echelon form the augmented matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
2 & 2 & 3 & 0 & 1 & 0 \\
3 & 4 & 2 & 0 & 0 & 1 \\
\end{bmatrix}
\]

By applying our fundamental row operations yields:

\[
\begin{align*}
-2 \cdot (\text{row 1}) + (\text{row 2}) & \rightarrow (\text{row 2}) : \\
-3 \cdot (\text{row 1}) + (\text{row 3}) & \rightarrow (\text{row 3}) : \\
\text{swap (row 2) and (row 3)} : & \\
(\text{row 3}) + (\text{row 2}) & \rightarrow (\text{row 2}) : \\
-1 \cdot (\text{row 3}) + (\text{row 1}) & \rightarrow (\text{row 1}) : \\
-1 \cdot (\text{row 2}) + (\text{row 1}) & \rightarrow (\text{row 1}) : \\
\end{align*}
\]

and thus

\[
A^{-1} = \begin{bmatrix}
8 & -2 & -1 \\
-5 & 1 & 1 \\
-2 & 1 & 0 \\
\end{bmatrix}.
\]