1. (30 points) The following TRUE/FALSE questions are unrelated. If the statement is always true write TRUE, otherwise write FALSE. Supporting work or justification is not necessary and will not be considered or graded.

(a) An eigenvalue with algebraic multiplicity three has an eigenspace with dimension three.

FALSE

(b) Consider the $n \times n$ matrix $A$. If $Ax = 0$ has only the solution $x = 0$, then $\lambda = 0$ is not an eigenvalue of $A$.

TRUE. Since $A$ is invertible (trivial nullspace).

(c) The general solution of $y'' - 2y' + y = x^2$ is given by $y(x) = C_1e^x + C_2xe^x + x^2 + 4x + 1$.

FALSE: Check that $y_p(x) = x^2 + 4x + 1$ does not satisfy $y'' - 2y' + y = x^2$.

(d) The functions $(1 - x), (1 + x), (1 - 3x)$ form a linearly independent set of solutions to $y'' = 0$ on $\mathbb{R}$.

FALSE: The functions are linearly dependent.

(e) If a $2 \times 2$ matrix $A$ has an eigenvalue 0 with multiplicity two, and the associated eigenspace is one-dimensional, then the solutions to the linear system $\mathbf{x}' = Ax$ are always bounded.

FALSE

(f) If $x_1(t)$ and $x_2(t)$ are solutions of $x'(t) + p(t)x = f(t)$, then $x(t) = x_1(t) + C(x_1(t) - x_2(t))$ with $C \in \mathbb{R}$ is the general solution of $x'(t) + p(t)x = f(t)$.

FALSE: We might have $x_1 \equiv x_2$.

2. (40 points) The following SHORT ANSWER questions are unrelated. For the questions in this problem, no motivation or supporting work is required. If you do submit supporting work, then box your answer, and know that your supporting work will not be graded.

(a) Find the general solution to $\dot{y} + (1 + t)y = 0$.

SOLUTION: $y(t) = ce^{t - t^2/2}, -\infty < c < \infty$.

(b) Determine a particular solution to the ODE $y'' + y = \sin(x)$ using the method of undetermined coefficients.

SOLUTION: The homogeneous equation $y'' + y = 0$ has solution $A\cos(x) + B\sin(x)$. Take $y_p(x) = Ax \cos(x) + Bx \sin(x)$. Upon plugging in, we find that $y_p'' + y_p = \sin(x)$ gives $-2A\sin(x) + 2B\cos(x) = \sin(x)$ so that $-2A = 1$ and $A = -\frac{1}{2}$ and $B = 0$. Hence, $y_p(x) = -\frac{1}{2}x \cos(x)$.

(c) What is the largest interval in $t$ on which the initial value problem

$$\sin(\pi t) y''(t) + \exp(t) y'(t) + \cos^{-1}(\pi t) y(t) = 0$$

with $y(0.8) = 0$, and $y'(0.8) = 1$ is guaranteed to have a unique solution?

SOLUTION: $t \in (0.5, 1)$.

(d) Suppose a $2 \times 2$ matrix $A$ has an eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with an associated eigenvalue $\lambda_1 = 3$, and an eigenvector $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with an associated eigenvalue $\lambda_2 = -2$. Compute $Av_3$ where $v_3 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

SOLUTION: $\begin{pmatrix} 9 \\ 4 \end{pmatrix}$.

(e) Determine and classify all equilibrium solutions for the differential equation $y' = -3(1 - y)y$.

SOLUTION: $y = 0, 1$, asymptotic stable at $y = 0$ and unstable at $y = 1$.

(f) For what initial conditions $y(t_0) = y_0$ is the ODE $y' = -y\sqrt{1 - y}$ guaranteed to have a unique solution?

SOLUTION: $y_0 \neq 1$. 
3. (20 points) You have a large, usually happy, colony of European honey bees \((Apis mellifera)\) in your backyard. Let \(p(t)\) represent the colony population as a function of time, \(t\), where time is measured in days. For each part below, write down a differential equation for \(p(t)\) that models the dynamics of the population under the assumptions stated. Be sure to identify any restrictions on parameters that you introduce. The description in each part is independent of the other parts. **Note:** you are **not** being asked to solve the differential equations.

(a) Suppose that the increase in the bee population is directly proportional to the current number of bees.

(b) Suppose that the bee population always doubles every 10 days.

(c) Suppose that the number of bees always tends toward the carrying capacity \(K > 0\) that the environment can sustain indefinitely.

(d) Suppose that the bee colony will become extinct if the population drops below the threshold value, \(T > 0\).

**Solution:**

(a) \[ \dot{p} = rp, \; r > 0. \]

(b) \[ \dot{p} = \frac{\ln 2}{10} p. \]

(c) \[ \dot{p} = rp(1 - p/K), \; r > 0. \]

(d) \[ \dot{p} = -rp(1 - p/T), \; r > 0. \]

4. (20 points) Consider the ODE \(y' = -6(y + 1)^{2/3}(3t^2 + 1)\).

(a) Find the general solution for \(y > -1\).

(b) Now, solve the initial value problem if \(y(-1) = 7\).

**Solution:**

(a) \[ \int \frac{1}{(y + 1)^{2/3}} dy = 3(y + 1)^{1/3} = \int -6(3t^2 + 1) dt = -6(t^3 + t) - 3C, \; y + 1 = -(2t^3 + 2t + C)^3, \; y = -(2t^3 + 2t + C)^3 - 1 \]

where \(C \in \mathbb{R}\).

(b) \(8 = C^3 \rightarrow c = 2\) and hence \(y = -(2t^3 + 2t + 2)^3 - 1\).

5. (30 points) Consider the linear system \(Ax = b\) where

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & k & 3 \\
3 & 5 & 7
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}
0 \\
0 \\
p
\end{pmatrix}.
\]

(a) For what value(s) of \(k\) and \(p\) does this equation have

i. A unique solution?
ii. Infinite number of solutions?
iii. No solution?

(b) Compute the determinant of \(A\). For what value(s) of \(k\) is \(A\) non-invertible?

(c) For what value(s) of \(k\) and \(p\) is the vector \(b\) NOT in the span of the columns of \(A\)? For what value(s) of \(k\) are the columns of \(A\) linearly independent?

**Solution:**

(a) i. \(k \neq 2;\)
ii. \(k = 2\) and \(p = 0;\)
iii. \(k = 2\) and \(p \neq 0.\)

(b) \(\det A = 4k - 8. \; k = 2.\)

(c) \(k = 2\) and \(p \neq 0. \; k \neq 2.\)
6. (30 points) A point mass $m$ is suspended from massless string of length $L > 0$. Let $\theta(t)$ be the angle of swing measured from vertical at time $t$. Also, let $g > 0$ be the acceleration of gravity. When at rest, the mass hangs straight down in which case $\theta = 0$. If disturbed from its vertical resting position, the pendulum will oscillate. For small angles of swing ($|\theta| \ll 1$) the motion of the pendulum can be approximated by the ODE

$$L \ddot{\theta} + g \theta = 0.$$  \hspace{1cm} (1)

(a) Determine the general solution of the second order ODE.
(b) Write an equivalent coupled system of linear first–order ODE’s equivalent to (1).
(c) Derive the general solution to the linear system in part (b)
(d) Identify and classify any equilibrium solution(s) of your linear system in part (b).

**Solution:**

(a) The corresponding characteristic equation is $r^2 + \frac{g}{L} = 0$. It follows that $r = \pm \sqrt{-\frac{g}{L}} = \pm i \sqrt{\frac{g}{L}}$. Thus, the general solution is given by:

$$\theta(t) = C_1 \cos(\sqrt{\frac{g}{L}} t) + C_2 \sin(\sqrt{\frac{g}{L}} t)$$

(b) Let $y = \dot{\theta}$. We obtain the system:

$$\begin{align*}
\theta' &= y \\
y' &= -\frac{g}{L} \theta
\end{align*}$$

which is equivalent to:

$$[\begin{bmatrix} \theta' \\ y \end{bmatrix}] = A [\begin{bmatrix} \theta \\ y \end{bmatrix}] = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} [\begin{bmatrix} \theta \\ y \end{bmatrix}]$$

We have $|A| = \frac{g}{L}$ and $Tr(A) = 0$, giving eigenvalues:

$$\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4|A|}}{2} = \pm i \sqrt{\frac{g}{L}}$$

For the eigenvectors, we solve $(A - \lambda_1 I)v_1 = 0$ with $\lambda_1 = -i \sqrt{\frac{g}{L}}$:

$$\begin{bmatrix} i \sqrt{\frac{g}{L}} & 1 \\ -\frac{g}{L} & i \sqrt{\frac{g}{L}} \end{bmatrix} [\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}] = [\begin{bmatrix} 0 \\ 0 \end{bmatrix}]$$

Taking the first equation $i \sqrt{\frac{g}{L}} v_1 + v_2 = 0$ yields the eigenvector

$$v_1 = \begin{bmatrix} 1 \\ -i \sqrt{\frac{g}{L}} \end{bmatrix}$$

The second eigenvector is the conjugate of $v_1$. Thus the complex valued solution is:

$$\tilde{x}_{cmp}(t) = e^{\lambda_1 t}v_1(t) = e^{-i \sqrt{\frac{g}{L}} t} \begin{bmatrix} 1 \\ -i \sqrt{\frac{g}{L}} \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{\frac{g}{L}} t) - i \sin(\sqrt{\frac{g}{L}} t) \\ -i \sqrt{\frac{g}{L}} \sin(\sqrt{\frac{g}{L}} t) - \sqrt{\frac{g}{L}} \cos(\sqrt{\frac{g}{L}} t) \end{bmatrix}$$

yielding the general solution (as a linear combination of real and complex parts):

$$\tilde{x}(t) = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix} = \tilde{C}_1 \begin{bmatrix} \cos(\sqrt{\frac{g}{L}} t) \\ -\sqrt{\frac{g}{L}} \sin(\sqrt{\frac{g}{L}} t) \end{bmatrix} + \tilde{C}_2 \begin{bmatrix} -\sin(\sqrt{\frac{g}{L}} t) \\ -\sqrt{\frac{g}{L}} \cos(\sqrt{\frac{g}{L}} t) \end{bmatrix}$$

(c) The equilibrium point of

$$[\begin{bmatrix} \theta' \\ y \end{bmatrix}] = A [\begin{bmatrix} \theta \\ y \end{bmatrix}] = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} [\begin{bmatrix} \theta \\ y \end{bmatrix}]$$

is $(0,0)$. Since the eigenvalues are $\lambda_{1,2} = \pm i \sqrt{\frac{g}{L}}$ this corresponds to solution curves forming a pattern of stable centers (circles) around the origin. The stability means that when the mass bob is at rest facing straight down (when the deflection angle $\theta = 0$), a small force applied to the bob will keep the displacement of the bob close to the equilibrium position.
7. (30 points) Consider the coupled nonlinear system of ODEs for $x(t)$ and $y(t)$

\[
\begin{align*}
\frac{dx}{dt} &= x + x^2 - 3xy \\
\frac{dy}{dt} &= 3y - y^2 - xy.
\end{align*}
\]

(a) Determine the $(x(t), y(t))$ values of any equilibrium solution(s).

SOLUTION: $(0, 0), (-1, 0), (0, 3), (2, 1)$, which are the intersections of the four nullclines $x = 0$, $x - 3y = -1$, $y = 0$, and $x + y = 3$.

(b) Consider the equilibrium solution farthest from the origin in a phase portrait. Determine the linearization of the original nonlinear system at this location.

SOLUTION: The point $(0, 3)$ is farthest from the origin. Transform using $u(t) = x(t) - 0$ and $v(t) = y(t) - 3$, and then study the system

\[
\begin{bmatrix}
u'
\end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 + 2x - 3y & -3x \\ -y & 3 - 2y - x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

from which we see that $\text{Det}(J) = 24$ and $\text{Trace}(J) = -11$.

(c) Classify the equilibrium solution found in part (b).

SOLUTION: Attracting node (aka sink) and is asymptotically stable. Two distinct $\lambda$s, both real and negative.