Problem 1: (20 points) This problem will be graded solely on the answers provided. Motivations/derivations will not be considered.

(a) (5 points) For which of the parameters \( b \) and \( k \) does the differential equation

\[
y''(t) + by'(t) + ky(t) = 0 \tag{1}
\]

describe underdamped oscillations? Recall that an oscillator is underdamped if, as \( t \to \infty \), \( y(t) \) crosses 0 infinitely many times as it approaches \( y = 0 \).

(i) \( b = 4, k = 1 \);
(ii) \( b = 1, k = 1 \);
(iii) \( b = -1, k = 1 \);
(iv) \( b = 2, k = 2 \).

(b) (5 points) Consider the family of matrices

\[
A = \begin{bmatrix} 1 & 1 \\ t & 2 \end{bmatrix}, \quad t \in \mathbb{R}.
\]

For which values of \( t \) does \( A \)

(i) have distinct, real eigenvalues?
(ii) have a double eigenvalue?
(iii) have complex eigenvalues with nonzero imaginary part?

(c) (4 points) Is the set of solutions to the differential equation

\[
y''(t) + t^2 y(t) = t^3 y'(t) + ty'''(t) \tag{2}
\]

a vector space? If yes, what is its dimension?

(d) (6 points) What are the eigenvalues of the following matrices?

(i) \( \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \);
(ii) \( \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \);
(iii) \( \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \);

\[\textit{Solution:}\]

(a) First, we must have \( b \geq 0 \) and \( k > 0 \) for the differential equation to describe a harmonic oscillator. To have an underdamped oscillator, we further require the discriminant \( \Delta = b^2 - 4k < 0 \). These conditions are satisfied for [ii, iv].

(b) The characteristic polynomial is

\[
\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - t = \lambda^2 - 3\lambda + 2 - t. \tag{3}
\]
The discriminant is \( \Delta = 9 - 4(2 - t) = 1 + 4t \). Hence (i) \( \Delta > 0 \Rightarrow t > -1/4 \); (ii) \( \Delta = 0 \Rightarrow t = -1/4 \); (iii) \( \Delta < 0 \Rightarrow t < -1/4 \).

(c) This differential equation is linear and homogeneous, so the set of solutions form a vector space whose dimension is the order of the differential equation. Therefore the answer is Yes, 3.

(d) (i) \( \det(A - \lambda I) = (1 - \lambda)(-1 - \lambda) - (-1) = \lambda^2 = 0 \Rightarrow \lambda = 0 \); (ii) \( \det(A - \lambda I) = (1 - \lambda)(-1 - \lambda) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \); (iii) \( \det(A - \lambda I) = (1 - \lambda)(-1 - \lambda) - 1 = \lambda^2 - 2 = 0 \Rightarrow \lambda = \pm \sqrt{2} \).

Problem 2: (20 points) Consider the differential equation
\[ y'' - \frac{2}{t^2}y = -3, \]
for \( t \geq 1 \) with homogeneous solutions \( y_1(t) = t^2 \), \( y_2(t) = 1/t \). Use the method of variation of parameters to find the general solution to the nonhomogeneous problem.

Solution:
To apply the variation of parameters method, we seek a solution in the form
\[ y(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = v_1(t)t^2 + \frac{v_2(t)}{t}. \]
Upon insertion of this guess into the nonhomogeneous differential equation and adding an additional constraint equation, we can uniquely determine \( v_1(t) \) and \( v_2(t) \) up to an integral
\[ v'_1(t) = -\frac{y_2(t)f(t)}{W(t)}, \quad v'_2(t) = \frac{y_1(t)f(t)}{W(t)}, \]
where \( f(t) = -3 \) is the inhomogeneity and \( W(t) \) is the Wronskian of the two, linearly independent homogeneous solutions
\[ W = \begin{vmatrix} t^2 & 1/t \\ 2t & -1/t^2 \end{vmatrix} = -1 - 2 = -3. \]
Then the solutions for \( v_i(t) \) are
\[ v_1(t) = \int \left( -\frac{1}{t} \right) dt = -\ln t, \]
\[ v_2(t) = \int t^2 dt = \frac{1}{3} t^3. \]
Utilizing the nonhomogeneous principle, the general solution is the homogeneous solution plus a particular solution:
\[ y(t) = c_1 t^2 + c_2/t - t^2 \ln t + \frac{1}{3} t^2, \quad c_1, c_2 \in \mathbb{R}. \]

Problem 3: (20 points) Consider the differential equation
\[ y'' + 4y' + 5y = e^{-2t}. \]

(a) (10 points) Find the general solution to the corresponding homogeneous equation.
(b) (10 points) Use the method of undetermined coefficients to find a particular solution to the nonhomogeneous problem.

Solution:
(a) We seek the general solution $y_h(t)$ to
\[ y'' + 4y' + 5y = 0. \]
This is a constant coefficient, linear equation so we look for solutions of the form $y_h(t) = e^{rt}$. Inserting this guess into the homogeneous equation gives
\[ r^2e^{rt} + 4re^{rt} + 5e^{rt} = 0 \quad \Rightarrow \quad r^2 + 4r + 5 = 0. \]
The roots of the characteristic equation are $r_{\pm} = -2 \pm i$ so the general homogeneous solution can be written
\[ y_h(t) = e^{-2t}(c_1 \cos t + c_2 \sin t), \quad c_1, c_2 \in \mathbb{R}. \]

(b) We guess a particular solution in the form $y_p(t) = Ae^{-2t}$. Inserting this into the nonhomogeneous equation gives
\[ 4Ae^{-2t} - 8Ae^{-2t} + 5Ae^{-2t} = e^{-2t} \quad \Rightarrow \quad 4A - 8A + 5A = A = 1. \]
Then a particular solution is $y_p(t) = e^{-2t}$.

**Problem 4:** (20 points)

(a) (16 points) Solve the initial value problem
\[
\begin{align*}
\dot{y} + 9y &= \sin(3t), \\
y(0) &= 0, \\
\dot{y}(0) &= 1.
\end{align*}
\]
(b) (4 points) Find one solution to the differential equation
\[ \dot{y} + 9y = \sin(3t - 1). \]

*Hint: There is a relatively quick way to solve this Problem 4(b). Be mindful that this is worth 5 points and could be a lot of work.*

**Solution:**

(a) The homogeneous equation $\ddot{y} + 9y = 0$ has the general solution
\[ y_h(t) = A \cos(3t) + B \sin(3t). \]
To find a particular solution, we use the method of undetermined coefficients. Since $\sin(3t)$ is a solution of the homogeneous equation, we try
\[ y_p = Ct \cos(3t) + Dt \sin(3t). \]
Inserting our “guess” into the differential equation, we find
\[
\begin{align*}
(-6C \sin(3t) - 9Ct \cos(3t) + 6D \cos(3t) - 9Dt \sin(3t)) + 9(Ct \cos(3t) + Dt \sin(3t)) &= \sin(3t),
\end{align*}
\]
which simplifies to
\[ -6C \sin(3t) + 6D \cos(3t) = \sin(3t). \]
The solution is $C = -1/6$ and $D = 0$, so
\[ y_p = -\frac{1}{6}t \cos(3t). \]
The general solution is then
\[ y = y_h + y_p = A \cos(3t) + B \sin(3t) - \frac{1}{6}t \cos(3t). \]
Inserting the initial data, we find
\[
\begin{align*}
0 &= y(0) = A, \\
1 &= \dot{y}(0) = 3B - 1/6.
\end{align*}
\]
The solution is $A = 0$ and $B = 7/18$, so we get the final solution
\[ y = \frac{7}{18} \sin(3t) - \frac{1}{6} t \cos(3t). \]

(b) Let $f(t) = \sin(3t)$ denote the forcing in part (a). Then the forcing in this part is $\sin(3(t - 1/3)) = f(t - 1/3)$. In other words, the new forcing is the same as the one in part (a), just shifted by $1/3$. So we obtain one solution simply by shifting the particular solution we found in (a),
\[ y = -\frac{1}{6} (t - \frac{1}{3}) \cos(3t - 1). \]

Note that since we can write this solution as
\[ y = -\frac{1}{6} t \cos(3t - 1) + \frac{1}{18} \cos(3t - 1), \]
we observe that the second term is just a homogeneous solution. So, another particular solution is
\[ y = -\frac{1}{6} t \cos(3t - 1). \]

Problem 5: (20 points) Find all eigenvalues and eigenvectors to the following two matrices
\[ A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}. \]

Solution:
For matrix $A$, we find the characteristic equation
\[ 0 = \det \begin{bmatrix} -3 - \lambda & 4 \\ 4 & 3 - \lambda \end{bmatrix} = (-3 - \lambda)(3 - \lambda) - 16 = \lambda^2 - 25. \]

The solutions are $\lambda_1 = 5$ and $\lambda_2 = -5$

Find evects for $\lambda_1 = 5$: We need to solve $(A - 5I)v = 0$, or
\[ \begin{bmatrix} -8 & 4 & 0 \\ 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} -8 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

The set of all solutions is $v_1 \in \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Find evects for $\lambda_2 = -5$: We need to solve $(A + 5I)v = 0$, or
\[ \begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

The set of all solutions is $v_2 \in \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

To process matrix $B$, we observe that since $B$ is triangular, it has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

Find evects for $\lambda_1 = 1$: We need to solve $(B - I)v = 0$, or
\[ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]
There is only one free parameter, so despite the fact that 1 is a double root, there is only one linearly independent eigenvector. The set of all these eigenvectors is \( \mathbf{v}_1 \in \text{span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \).

**Find evecs for \( \lambda_2 = 2 \):** We need to solve \((\mathbf{B} - 2I)\mathbf{v} = 0\), or
\[
\begin{bmatrix}
-1 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

There is only one free parameter, and we find the set of all eigenvectors is \( \mathbf{v}_2 \in \text{span}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \).