Problem 1: (20 points) For the questions in this problem, no motivation is required. If you do submit work, then box your answer, and know that your work will not be graded.

(a) (5p) Let $A$ and $B$ be $3 \times 3$ matrices, and consider the following four matrices:
\[
C = A^2 + 2AB + B^2, \\
D = A(A + B) + B(B + A), \\
E = (A + B)(B + A), \\
F = A^2 + AB + BA + B^2.
\]
Which of these matrices must necessarily equal $(A + B)^2$?

(b) (5p) Let $k$ be a real number, and consider the matrix
\[
A = \begin{pmatrix} k & 2 \\ 6 & k - 1 \end{pmatrix}.
\]
For which $k$ is the rank of $A$ less than 2?

(c) (5p) Let $v_1$, $v_2$, and $v_3$ be three vectors in $\mathbb{R}^3$. You know that $V = \text{span}(v_1, v_2)$ is a plane through the origin (its dimension is two), and that $W = \text{span}(v_3)$ is a line through the origin. Then $\text{span}(v_1, v_2, v_3)$ is the smallest vector space containing both $V$ and $W$. What are the possible values of the dimension of $\text{span}(v_1, v_2, v_3)$?

(d) (5p) Let $V$ denote the vector space of continuous functions on an interval $[-1, 1]$. Which of the following subsets of $V$ are vector spaces?

(i) The set $W_1$ of all even functions in $V$.
(ii) The set $W_2$ of all odd functions in $V$.
(iii) The set $W_3$ of all functions $f$ in $V$ such that $f(1) = 0$.
(iv) The set $W_4$ of all functions $f$ in $V$ such that $f(0) = 1$.

Solution:

(a) $D$, $E$, and $F$.

(b) For $k = 4$ or $k = -3$.

(c) The dimension can be two or three.

(d) $W_1$, $W_2$, and $W_3$ are vector spaces.

Motivations are not required, but here are some comments, for clarity:

(a) Note that $(A + B)^2 = A^2 + AB + BA + B^2$. In general, it is not necessarily the case that $AB = BA$.

(b) $\det(A) = k(k - 1) - 12 = k^2 - k - 12 = (k - 4)(k + 3)$ so $\det(A) = 0$ if $k = 4$ or $k = -3$.

(c) The dimension is two if the vector $v_3$ lies in the plane spanned by $v_1$ and $v_2$. Otherwise, the three vectors are linearly independent.

(d) Observe that $W_4$ is not a vector space since, for instance, $0 \notin W_4$. 
Problem 2: (20 points) Compute the determinant and the inverse of the following matrix:

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 1 \\
1 & 0 & 2
\end{bmatrix}.
\]

**Solution:**

Using the standard rule for $3 \times 3$ determinants, we get

\[
\det(A) = 4 + 0 + 0 - 2 - 0 - 0 = 2.
\]

For the inverse, we apply row operations to the extended system

\[
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & 0 & 1
\end{bmatrix} \sim \{R_3^* = R_3 - R_1\} \sim \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 1
\end{bmatrix} \sim \{R_2^* = R_2 - R_3, \ R_1^* = R_1 - R_3\}
\]

\[
\sim \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & -1 \\
0 & 2 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 1
\end{bmatrix} \sim \{R_2^* = \frac{1}{2} R_2\} \sim \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & -1 \\
0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\
0 & 0 & 1 & -1 & 0 & 1
\end{bmatrix}.
\]

Problem 3: (20 points) Find all solutions to the linear system

\[
\begin{align*}
x_1 + 2x_2 + x_3 + 3x_4 &= 1, \\
x_1 + 2x_2 + 2x_3 + 5x_4 &= 0, \\
-x_1 - 2x_2 - 3x_3 - 7x_4 &= 1.
\end{align*}
\]

**Solution:**

Form the extended system matrix and perform row operations

\[
\begin{bmatrix}
1 & 2 & 1 & 3 & 1 \\
1 & 2 & 2 & 5 & 0 \\
-1 & -2 & -3 & -7 & 1
\end{bmatrix} \sim \{R_2^* = R_2 - R_1 \ \text{and} \ \ R_3^* = R_3 + R_1\}
\]

\[
\sim \begin{bmatrix}
1 & 2 & 1 & 3 & 1 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & -2 & -4 & 2
\end{bmatrix} \sim \{R_3^* = R_3 + 2R_2\}
\]

\[
\sim \begin{bmatrix}
1 & 2 & 1 & 3 & 1 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \sim \{R_1^* = R_1 - R_2\} \sim \begin{bmatrix}
1 & 2 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We see that $x_2$ and $x_4$ are “free” parameters. Set $x_2 = s$ and $x_4 = t$. Then the solutions are

\[
\begin{align*}
x_1 &= 2 - 2s - t \\
x_2 &= s \\
x_3 &= -1 - 2t \\
x_4 &= t.
\end{align*}
\]
Problem 4: (20 points) Consider the matrices

\[
A = \begin{bmatrix}
0 & 2 & 1 & 2 \\
1 & -4 & -1 & -3 \\
-1 & 6 & 2 & 5
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 2 \\
1 & -4 \\
-1 & 6
\end{bmatrix}.
\]

Hint: The matrix \( A \) is row-equivalent to the matrix \( C = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1/2 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \).

(a) (8p) Which of the following four sets are linearly independent:

(i) The columns of \( A \).
(ii) The columns of \( B \).
(iii) The rows of \( A \).
(iv) The set \( \begin{bmatrix}
2 \\
-4 \\
6
\end{bmatrix}, \begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix}, \begin{bmatrix}
-3 \\
2 \\
5
\end{bmatrix} \) consisting of the last three columns of \( A \).

(b) (8p) Recall that \( \text{col}(A) = \{ y \in \mathbb{R}^3 \text{ such that } y = Ax \text{ for some } x \in \mathbb{R}^4 \} \) is the span of the column vectors of \( A \), and that \( \text{null}(A) = \{ x \in \mathbb{R}^4 \text{ such that } Ax = 0 \} \) is the set of all solutions to \( Ax = 0 \).

What are the dimensions of \( \text{col}(A) \) and \( \text{null}(A) \)?

(c) (4p) Specify a basis for \( \text{null}(A) \).

Solution:

(a) (i) No, the columns of \( A \) are not linearly independent. There are many arguments for this. For example, there are four columns but each column is an element of \( \mathbb{R}^3 \) which is three dimensional, i.e., there could be at most three linearly independent vectors.

(ii) Yes, the two columns of \( B \) are linearly independent because the rank of \( B \) is two (from \( C \)).

(iii) No, the three rows of \( A \) are not linearly independent because the rank of \( A \) is two (from the RREF \( C \)).

(iv) No, the last three columns of \( A \) are not linearly independent because the rank of \( A \) is two.

(b) The RREF of \( A, C \), demonstrates that there are only two linearly independent columns of \( A \) so \( \dim \text{col}(A) = 2 \).

Since \( C \) is row equivalent to \( A \), \( \text{null}(A) = \text{null}(C) \). The first two rows of \( C \) place two restrictions on an element \( x \in \text{null}(C) \):

\[
\begin{align*}
  x_1 + x_3 + x_4 &= 0 \\
  x_2 + \frac{1}{2}x_3 + x_4 &= 0
\end{align*}
\]

So, \( x_1 \) and \( x_2 \) are determined by \( x_3 \) and \( x_4 \). Since \( x_3 \) and \( x_4 \) are arbitrary, there are two degrees of freedom and \( \dim \text{null}(A) = \dim \text{null}(C) = 2 \).

(c) Solving equations (2) and (3) for \( x_1 \) and \( x_2 \) gives

\[
x_1 = -x_3 - x_4, \quad x_2 = -\frac{1}{2}x_3 - x_4.
\]

Therefore, every element \( x \in \text{null}(A) \) has the form

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-x_3 - x_4 \\
-\frac{1}{2}x_3 - x_4 \\
x_3 \\
x_4
\end{bmatrix} = x_3 \begin{bmatrix}
-1 \\
-\frac{1}{2} \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}, \text{ for some } x_3, x_4 \in \mathbb{R}.
\]

Then a basis for \( \text{null}(A) \) is

\[
\left\{ \begin{bmatrix}
-1 \\
-\frac{1}{2} \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
1 \\
0 \\
1
\end{bmatrix} \right\}.
\]
**Problem 5:** (20 points) For the system of differential equations

\[
\frac{dx}{dt} = x (y - 1)
\]
\[
\frac{dy}{dt} = y (y + x^2 - 2),
\]
provide the following qualitative analysis in the phase plane for the region \(0 \leq x \leq 2\) and \(0 \leq y \leq 2\):

(a) (4p) Determine and sketch the nullclines for this system.

(b) (4p) What are the equilibrium solutions?

(c) (4p) Show the direction of trajectories on the nullclines.

(d) (4p) Classify the stability or instability of the equilibrium solutions.

(e) (4p) Specify the long term behavior of the solution starting at \(x_0 = 0.5\) and \(y_0 = 0.5\).

---

**Solution:**

(a) (4p) \(dx/dt = 0\) nullclines: \(x = 0\), \(y = 1\).
\(dy/dt = 0\) nullclines: \(y = 0\), \(y = 2 - x^2\).

(b) There are three equilibria in the region of interest: \((x, y) \in \{(0, 0), (0, 2), (1, 1)\}\). An additional equilibrium \((-1, 1)\) is outside the region of interest.

(c) \(x = 0\): \(dy/dt = y(y - 2)\)
\(y = 1\): \(dy/dt = x^2 - 1\)
\(y = 0\): \(dx/dt = -x\)
\(y = 2 - x^2\): \(dx/dt = x(1 - x^2)\)
See figure 1 for the slope directions.

(d) \((x, y) = (0, 0)\): stable
\((x, y) = (0, 2)\): unstable
\((x, y) = (1, 1)\): unstable

(e) For this initial condition, based on the phase plane analysis,

\[
\lim_{t \to \infty} x(t) = 0, \quad \lim_{t \to \infty} y(t) = 0.
\]
This initial condition is inside the basin of attraction of the equilibrium at \((0, 0)\).

---

**Figure 1.** Phase plane sketch for problem 5.