Problem 1: (30 points) Determine whether the following statements are TRUE or FALSE. (5 points each.).

(a) The differential equation \( y' + \frac{3+t}{2+\sin t} y = \frac{t^2 y}{2+\sin t} \) is linear.
(b) The differential equation \( y' + \frac{3+t}{2+\sin t} y = \frac{t^2 y}{2+\sin t} \) is separable.
(c) The approximate solution to \( y' = f(t,y) \) obtained using Euler’s method is a solution if \( f(t,y) = 2013 \).
(d) The linear operator \( L \) is defined by \( L[y] = y'' + p(t)y \). If \( y_1 \) is a solution of \( L[y] = 0 \), and \( y_2 = 2y_1 + 1 \) is a solution of \( L[y] = \cos t \) then we must have \( L[1] = \cos t \) and consequently \( p(t) = \cos t \).
(e) If \( y(t) \) is a solution to an autonomous DE \( (y' = f(y)) \), then there are \( t_1, t_2 \) where \( t_1 < t_2 \) such that \( y(t_1) = y(t_2) \) and \( y'(t_1) \neq y'(t_2) \).
(f) For some continuous function \( f \), the solution to the autonomous differential equation \( y' = f(y) \) could be \( y(t) = \cos(t) \).

Solution:

(a) True.
(b) True, \( y' = \frac{t^2-t-2}{2+\sin t} y \).
(c) True, The slope never changes, the solution is \( y = y_0 + 2013(t - t_0) \).
(d) True, \( L[1] = L[2y_1 + 1 - 2y_1] = L[2y_1 + 1] - 2L[y_1] = L[y_2] - 0 = \cos t \), also \( L[1] = 0 + p(t) = p(t) = \cos t \).
(e) False, this system is autonomous and therefore wherever \( y(t_1) = y(t_2) \) we know that \( y'(t_1) = y'(t_2) \).
(f) False, The solution to a 1D autonomous differential equation can never oscillate. If the solution oscillates, this means that isolines would need to depened on time and thus the differential equation would not be autonomous.

Problem 2: (30 points) Consider the differential equation

\[
(t + 1)^{2/3} \frac{dy}{dt} = y^{1/3}. \tag{1}
\]

(a) Solve Eq. (1) given the initial condition \( y(t = 0) = 2\sqrt{2} \).
(b) What does Picard’s theorem imply about the existence and uniqueness of solutions to Eq. (1) near (i) \((t, y) = (-1, 1)\); (ii) \((t, y) = (0, 0)\); (iii) \((t, y) = (0, 1)\)?

Solution:

(a) The only equilibrium is \( y = 0 \). If \( y \neq 0 \), we can use separation of variables to rewrite the ODE as

\[
(t + 1)^{-2/3} dt = y^{-1/3} dy. \tag{2}
\]

Integrating both sides

\[
3(t + 1)^{1/3} = \frac{3}{2} y^{2/3} + c. \tag{3}
\]

Using the initial condition

\[
3 = \frac{3}{2} \cdot 2 + c \quad \Rightarrow \quad c = 0. \tag{4}
\]
Therefore the solution to the IVP is given by

\[ 3(t + 1)^{1/3} = \frac{3}{2} y^{2/3} \Rightarrow y = 2\sqrt{2(t + 1)}. \]  

(5)

(b) To apply Picard’s theorem, we test the continuity of the ODE right hand side \( f \) and its partial derivative \( \frac{\partial f}{\partial y} \), which are respectively

\[ f = (t + 1)^{-2/3} y^{1/3}, \]  

and

\[ \frac{\partial f}{\partial y} = \frac{1}{3} (t + 1)^{-2/3} y^{-2/3}. \]  

(6)

(7)

For (i), \( f \) is not continuous at \((t, y) = (-1, 1)\), so the solution may not exist and may be non-unique.

For (ii), \( f \) is continuous but \( \frac{\partial f}{\partial y} \) is not continuous at \((t, y) = (0, 0)\), so the solution exists but may be non-unique.

For (iii), both \( f \) and \( \frac{\partial f}{\partial y} \) are continuous at \((t, y) = (0, 1)\), so the solution exists and is unique.

**Problem 3:** (30 points) Consider the initial value problem: \( y' + (3t^2 + \frac{1}{t})y = t, \quad y(1) = \frac{4}{3}. \)

(a) Determine an integrating factor for the differential equation.

(b) Determine the general solution to the differential equation.

(c) Determine the solution to the initial value problem.

**Solution:**

(a) The integrating factor is:

\[ \mu = e^{\int (3t^2 + \frac{1}{t})dt} = e^{t^3 + \ln t} = te^{t^3} \]

(b) The general solution to the differential equation is:

\[ te^{t^3} y' + (3t^2 + \frac{1}{t})te^{t^3} y = t^2 e^{t^3} \]

\[ \frac{d}{dt}[te^{t^3} y] = t^2 e^{t^3} \]

\[ te^{t^3} y = \int t^2 e^{t^3} dt = \frac{1}{3} e^{t^3} + C \]

\[ y = \frac{1}{3t} + \frac{C e^{-t^3}}{t} \]

(c) The solution to the initial value problem is:

\[ y(1) = \frac{4}{3} = \frac{1}{3} + \frac{C e^{-1}}{1} \Rightarrow C = e \]

\[ y(t) = \frac{1}{t} \left( \frac{1}{3} + e^{(1-t^3)} \right) \]

**Problem 4:** (30 points) Suppose that we have a 400-mg sample of certain radioactive material at time \( t_0 = 0 \) and that the half-life of the material is 30 years.

(a) Set up the corresponding initial-value problem for this decay problem.

(b) Find the mass of the materia after \( t \) years.

(c) How much of the sample remains after 150 years?
(d) After how long will only 100 mg remain?

Solution:

(a) \( y_0 = 400, k = \frac{\ln 2}{30} \), The IVP is: \( y' = -\frac{\ln 2}{30} y \), \( y(0) = 400 \).

(b) \( y(t) = 400e^{-\frac{\ln 2}{30} t} \).

(c) \( y(150) = 400e^{-\frac{\ln 2}{30} \cdot 150} = 400e^{-5 \ln 2} = 400e^{\ln \left(\frac{1}{2^5}\right)} = \frac{25 \times 2^4}{2^5} = \frac{25}{2} = 12.5 \text{mg} \).

(d) \( y(T) = 400e^{-\frac{\ln 2}{30} T} = 100, \ 4 = e^{\ln \frac{100}{25}}, \ \frac{\ln 2}{30} T = \ln 4, \ T = \frac{30 \ln 4}{\ln 2} = 60 \).

Problem 5: (30 points) Consider a situation in which Prof. Bortz’s basement is flooding with water. Let’s assume that his basement currently contains 1000L of water with cracks in the foundation causing an inflow of 6 L/min. Let’s also assume that at time \( t = 0 \), the groundwater breaks into a vein of gold and the water entering the basement now does so with a concentration of 5 milligrams of gold/Liter.

(a) Suppose Prof. Bortz’s kind neighbor lends him a fish tank pump that can send 6 L/min of water out of the basement through a hose. Assuming instantaneous mixing (due to Prof. Bortz wading around the basement in boots), what is the initial value problem that would model the amount of gold \( x(t) \) in the basement?

(b) Solve the above initial value problem for the amount of gold in the basement \( x(t) \).

(c) Suppose that after a few hours, the rain slows down and the rate of water entering the basement decreases to 3 L/min. How does this change the differential equation? Using any method at your disposal, solve for the general form of the solution to this new model for the amount of gold in the basement \( x(t) \).

Solution:

(a) The initial value problem is:

\[
\begin{align*}
\dot{x}(t) &= 30 - \frac{6}{1000} x(t) \\
x(0) &= 0
\end{align*}
\]

(b) By Euler-Lagrange Two-Step:

(i) Homogeneous Solution:

The homogeneous equation is

\[ \dot{x}'(t) = -\frac{6}{1000} x(t) \]

with a solution of

\[ x_h(t) = C e^{-\frac{3}{500} t} \]

(ii) Particular Solution:

Using Variation of Parameters, we propose a solution of the form

\[ x_p(t) = v(t) e^{-\frac{3}{500} t} \]

and by substituting into the original equation, obtain

\[
\begin{align*}
\dot{v}'(t) e^{-\frac{3}{500} t} &= 30 \\
v(t) &= 5000e^{\frac{3}{500} t}.
\end{align*}
\]
Therefore, we have that
\[ x_p(t) = 5000. \]

The general solution is thus
\[ x(t) = Ce^{-\frac{3}{500}t} + 5000 \]

and applying the initial condition of \( x(0) = 0 \) yields
\[
\boxed{x(t) = 5000(1 - e^{-\frac{3}{500}t})}
\]

(c) The new differential equation is
\[
x'(t) = 3 \cdot 5 - \frac{6}{1000 - 3t}x(t)
\]

This equation could be solved using an integrating factor or by Euler-Lagrange Two-Step. Here we will use the Integrating Factor method. The factor would be
\[
\mu(t) = e^{\int \frac{6}{1000 - 3t} dt} = (1000 - 3t)^{-2}
\]

and the general solution will be
\[
\boxed{x(t) = C(1000 - 3t)^2 + 5(1000 - 3t)}
\]