Problem 1: (20 points) Consider the linear differential operator
\[ L(y) = y'' - \frac{2}{t} y' + \frac{2}{t^2} y \]
(a) Verify that \( y(t) = t \) and \( y(t) = t^2 \) are both solutions to the homogeneous equation \( L(y) = 0 \).
(b) Construct a particular solution to the nonhomogeneous equation \( L(y) = f \), where \( f(t) = t^3 \).
(c) Write down the general solution to the equation \( L(y) = f \).

Solution:
(a) 
\[ L(t) = (t'')' - \frac{2}{t} (t')' + \frac{2}{t^2} (t) = 0 \]
and
\[ L(t^2) = (t^2)'' - \frac{2}{t} (t^2)' + \frac{2}{t^2} (t^2) = 0 \]
(b) Variation of parameters: Two homogeneous solutions given by \( y_1 = t, y_2 = t^2 \).
\[ W[y_1, y_2] = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2. \]
\[ v_1' = -\frac{y_2 f}{W[y_1, y_2]} = -\frac{t^3}{t^2} = -t \]
\[ v_2' = \frac{y_1 f}{W[y_1, y_2]} = \frac{t}{t^2} = \frac{t}{t^2} \]
So,
\[ v_1 = \int -t^3 dt = -\frac{1}{4} t^4 \]
\[ v_2 = \int t^2 dt = \frac{1}{3} t^3 \]
Then,
\[ y_p(t) = y_1 v_1 + y_2 v_2 \]
\[ = t - \frac{1}{4} t^4 + \frac{1}{3} t^3 \]
\[ = \frac{t^5}{12} \]
(c) 
\[ y(t) = c_1 y_1 + c_2 y_2 + y_p(t) \]
\[ = c_1 t + c_2 t^2 + \frac{t^5}{12} \]

Problem 2: (15 points) True/False. If the statement is always true, mark true. Otherwise, mark false. You do not need to show your work. (Any work will not be graded.)
(a) There exists a real \( 2 \times 2 \) matrix, \( A \), with eigenvalues \( \lambda_1 = 1, \lambda_2 = i \).
(b) If $\lambda = 2$ is a repeated eigenvalue of multiplicity 2 for a matrix $A$, then the eigenspace associated with $\lambda$ has dimension 2.

(c) If an $n \times n$ matrix has $n$ distinct eigenvalues, then each eigenvalue has a corresponding one-dimensional eigenspace.

Solution:

(a) False. Complex eigenvalues of real matrices are always present in complex conjugate pairs.

(b) False. Not necessarily, e.g. $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

(c) True. If $\{\lambda_1, \ldots, \lambda_n\}$ is a set of distinct eigenvalues then a corresponding set of eigenvectors $\{v_1, \ldots, v_n\}$ is linearly independent. This is a maximally linearly independent set in $\mathbb{R}^n$ so if any one eigenvalue had two (or more) linearly independent eigenvectors, then one of them would be a linear combination of eigenvectors of other eigenspaces, which would be a contradiction.

Problem 3: (20 points) Consider the following differential equation,

$$x'' + 2x' + x = e^t$$

(a) Find the general solution $x(t)$.

(b) Solve the initial value problem with $x(0) = x'(0) = 0$.

Solution:

(a) Homogeneous solution:

$$x''_h + 2x'_h + x_h = 0$$

$$\Rightarrow x_h = c_1 e^{-t} + c_2 te^{-t}$$

Undetermined coefficients:

$$x_p = Ae^t$$

$$\Leftrightarrow \ (Ae^t)'' + 2(Ae^t)' + (Ae^t) = e^t$$

$$\Leftrightarrow \ A = \frac{1}{4}$$

General Solution:

$$x(t) = x_h(t) + x_p(t)$$

$$= c_1 e^{-t} + c_2 te^{-t} + \frac{1}{4} e^t.$$ 

(b)

$$x(0) = 0 \Rightarrow 0 = c_1 + \frac{1}{4} \Rightarrow c_1 = -\frac{1}{4}$$

$$x'(0) = 0 \Rightarrow 0 = -c_1 + c_2 + \frac{1}{4} \Rightarrow c_2 = -\frac{1}{2}$$

So,

$$x(t) = -\frac{1}{4} e^{-t} - \frac{1}{2} te^{-t} + \frac{1}{4} e^t.$$
Problem 4: (20 points) Find the general solution to the system of differential equations
\[
\ddot{x} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \dot{x}.
\]

Solution: \( p(\lambda) = (1 - \lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 \Rightarrow \lambda = 1 \pm 2i. \)
Find the eigenvector corresponding to \( \lambda = 1 + 2i \): Solve \((A - \lambda I)\vec{v} = \vec{0}\):
\[
\begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \end{pmatrix} \sim \begin{pmatrix} -2i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

So,
\[
\overline{x}_{\Re} = e^{\alpha t}(\cos \beta t \vec{p} - \sin \beta t \vec{q}) = e^t \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}
\]
\[
\overline{x}_{\Im} = e^{\alpha t}(\sin \beta t \vec{p} + \cos \beta t \vec{q}) = e^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.
\]

Then the general solution is,
\[
\overline{x}(t) = c_1 \overline{x}_{\Re} + c_2 \overline{x}_{\Im} = e^t (c_1 \cos 2t + c_2 \sin 2t) - c_1 \sin 2t + c_2 \cos 2t
\]

Problem 5: (25 points) Consider the matrix
\[
A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.
\]

(a) Find all the eigenvalues of \( A \).
(b) For each eigenvalue that you found in part (a), find a basis for the corresponding eigenspace.
(c) Find the general solution of \( \ddot{x} = A\dot{x} \). (Hint: you need to find a generalized eigenvector)

Solution:
(a) The characteristic polynomial of \( A \) is \( p(\lambda) = (1 - \lambda)(2 - \lambda)^2 \). The eigenvalues are \( \lambda_1 = 1, \lambda_2 = 2 \) (repeated).
(b) \( \lambda_1 = 1 \): Solve \((A - \lambda_1 I)\vec{v}_1 = \vec{0} \):
\[
\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
\]

Then \( \vec{v}_1 \) is a basis for the eigenspace \( \mathbb{E}_1 \) corresponding to \( \lambda_1 = 1 \).
(c) \( \lambda_2 = 2 \): Solve \((A - \lambda_2 I)\vec{v}_2 = \vec{0} \):
\[
\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

So \( \vec{v}_2 \) is a basis for the eigenspace \( \mathbb{E}_2 \) corresponding to \( \lambda_2 = 2 \).
(c) \( \ddot{x} = A\dot{x} \) is a \( 3 \times 3 \) system, so we need three linearly independent solutions. Two solutions are given by \( \vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 \) and \( \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 \). For the third, we need to find a generalized eigenvector for the eigenvalue \( \lambda_2 = 2 \), since this is the repeated eigenvalue.
Solve \((A - \lambda_2 I)\vec{u} = \vec{v}_2:\)

\[
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \vec{u} = s \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

for a free parameter \(s\). Since we only need one such vector, set \(s = 0\). Then a third linearly independent solution is given by

\[
\vec{x}_3(t) = t e^{\lambda_2 t} \vec{v}_2 + e^{\lambda_2 t} \vec{u}
\]

\[
= t e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.
\]

The general solution is given by

\[
\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
\]

\[
= c_1 e^t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ t \\ -1 \end{pmatrix}
\]

\[
= \begin{pmatrix}
-c_1 e^t + c_3 e^{2t} \\
c_1 e^t + c_2 e^{2t} + c_3 t e^{2t} \\
-c_3 e^{2t}
\end{pmatrix}.
\]