ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor’s name, (3) your recitation section number and (4) a grading table. Text books, class notes, and calculators are NOT permitted. A one page (2 sided, letter sized, handwritten) crib sheet is allowed.

Problem 1: (20 points) Consider the linear differential operator

\[ L(y) = y'' - \frac{2}{t} y' + \frac{2}{t^2} y \]

(a) Verify that \( y(t) = t \) and \( y(t) = t^2 \) are both solutions to the homogeneous equation \( L(y) = 0 \).
(b) Construct a particular solution to the nonhomogeneous equation \( L(y) = f \), where \( f(t) = t^3 \).
(c) Write down the general solution to the equation \( L(y) = f \).

Problem 2: (15 points) True/False. If the statement is always true, mark true. Otherwise, mark false. You do not need to show your work. (Any work will not be graded.)
(a) There exists a real \( 2 \times 2 \) matrix, \( A \), with eigenvalues \( \lambda_1 = 1 \), \( \lambda_2 = i \).
(b) If \( \lambda = 2 \) is a repeated eigenvalue of multiplicity 2 for a matrix \( A \), then the eigenspace associated with \( \lambda \) has dimension 2.
(c) If an \( n \times n \) matrix has \( n \) distinct eigenvalues, then each eigenvalue has a corresponding one-dimensional eigenspace.

Problem 3: (20 points) Consider the following differential equation,

\[ x'' + 2x' + x = e^t \]

(a) Find the general solution \( x(t) \).
(b) Solve the initial value problem with \( x(0) = x'(0) = 0 \).

Problem 4: (20 points) Find the general solution to the system of differential equations

\[ \vec{x}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \vec{x}. \]

Problem 5: (25 points) Consider the matrix

\[ A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \]

(a) Find all the eigenvalues of \( A \).
(b) For each eigenvalue that you found in part (a), find a basis for the corresponding eigenspace.
(c) Find the general solution of \( \vec{x}' = A\vec{x} \). (Hint: you need to find a generalized eigenvector)