On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number and instructor. This exam is worth 150 points and has 7 questions on both sides of this paper.

1. (28 points, 7 points each) Decide whether the following quantities are convergent or divergent. Explain your reasoning and name any test you use.

   (a) \[ \int_0^\infty \frac{\tan^{-1} x}{2 + e^x} \, dx \]

   (b) \[ \sum_{k=1}^\infty \frac{\sqrt{2}}{2^k} \]

   (c) \[ \frac{\pi}{3} + \frac{\pi}{6} + \frac{\pi}{9} + \frac{\pi}{12} + \frac{\pi}{15} + \cdots \]

   (d) The sequence given by \[ a_n = n \ln \left( \frac{n+1}{n} \right) \] for \( n = 1, 2, 3, \ldots \)

   **Solution:**

   (a) Since \( 0 < \frac{\tan^{-1} x}{2 + e^x} < \frac{\pi/2}{e^x} \) and \( \frac{\pi}{2} \int_0^\infty \frac{dx}{e^x} \) is convergent, by the Comparison Test \( \int_0^\infty \frac{\tan^{-1} x}{2 + e^x} \, dx \)

   also is convergent.

   (b) \[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} n \ln \left( 1 + \frac{1}{n} \right) = \lim_{n \to \infty} \ln \frac{1+1/n}{1/n} = \lim_{n \to \infty} \frac{1}{-1/n^2} = 1 \]

   The sequence is convergent.

   (c) The series diverges by the Test for Divergence because \( \lim_{k \to \infty} 2^{1/k} = 2^0 = 1 \neq 0 \).

   (d) \[ \frac{\pi}{3} + \frac{\pi}{6} + \frac{\pi}{9} + \cdots = \sum_{n=1}^\infty \frac{\pi}{3n} = \frac{\pi}{3} \sum_{n=1}^\infty \frac{1}{n} \]

   is a constant multiple of the divergent harmonic series and therefore also divergent.

2. (25 points) Consider the parametric curve given by \((x, y) = (2t^2, 1 + 12t - t^2)\).

   (a) Find the equation of the line with slope 1 that is tangent to the curve \((x, y) = (2t^2, 1 + 12t - t^2)\).

   (b) Compute \( \frac{d^2y}{dx^2} \) for this parametric curve. Simplify your answer.

   (c) Use the Trapezoidal Rule with \( n = 3 \) to estimate the area between the \( x \)-axis and the parametric curve from \( t = 1 \) to \( t = 2 \). Simplify your answer.

   (d) Estimate the error in part (c).

   **Solution:**

   (a) Note that

   \[ \text{slope} = \frac{dy}{dx} = \frac{12 - 2t}{4t} = 1 \implies 6t = 12 \implies t = 2 \implies (x, y) = (8, 21) \]

   and so the equation of the tangent line is

   \[ y - 21 = 1 \cdot (x - 8) \implies y = x + 13 \]

   **Alternate Solution:**

   Convert the equations to Cartesian form:

   \[ x = 2t^2 \implies t = \pm \sqrt{\frac{x}{2}} \implies y = 1 \pm 12 \sqrt{\frac{x}{2}} - \frac{x}{2} \implies \frac{dy}{dx} = \pm 3\sqrt{2} - \frac{1}{2} = 1 \implies (x, y) = (8, 21). \]
(b) In parametric form:

\[
\frac{d^2y}{dx^2} = \frac{4}{dt} \frac{dy/dx}{dx/dt} = \frac{1}{4t} \cdot \frac{4t(-2) - (12 - 2t)(4)}{16t^2} = -\frac{3}{4t^3}
\]

Alternate Solution:

In Cartesian form:

\[
\frac{dy}{dx} = \frac{3\sqrt{2}}{\sqrt{x}} - \frac{1}{2} \Rightarrow \frac{d^2y}{dx^2} = -\frac{3\sqrt{2}}{2x^{3/2}} = -\frac{3}{\sqrt{2}x^{3/2}}
\]

(c) Since \(x(1) = 2\) and \(x(2) = 8\), we are approximating the area under the curve on the \(x\)-interval \([2, 8]\).

For \(n = 3\), the subinterval length is \(\Delta x = 6/3 = 2\) and the partition points are \(x_i = 2, 4, 6, 8\), which correspond to \(t = 1, \sqrt{2}, \sqrt{3}, 2\), respectively.

\[
T_3 = \frac{\Delta x}{2} \left( y(1) + 2y \left(\sqrt{2}\right) + 2y \left(\sqrt{3}\right) + y(2) \right)
\]

\[
= \frac{2}{2} \left( 1 + 12 - 1 \right) + 2 \left( 1 + 12\sqrt{2} - 2 \right) + 2 \left( 1 + 12\sqrt{3} - 3 \right) + (1 + 24 - 4)
\]

\[
= 27 + 24\sqrt{2} + 24\sqrt{3}
\]

(d) For \(1 \leq t \leq 2\), the second derivative \(\left| \frac{d^2y}{dx^2} \right| = 3/(4t^3)\) has a maximum value of \(3/4\) at \(t = 1\).

Let \(K = 3/4\) and \(b - a = 6\). Then

\[
|E_T| \leq \frac{K(b - a)^3}{12n^2} = \frac{(3/4)(6^3)}{12(3^2)} = \frac{3}{2}
\]

3. (20 points) Consider the polar curve given by \(r^2 = \sin(2\theta)\) and use it to answer the following questions.

(a) Sketch the graph of this equation.

(b) Find the area of the region enclosed by the polar curve.

(c) Set up, but don’t evaluate, the integral to calculate the length of this polar curve.

Solution:

(b) Here we have one loop is parametrized by \(r = \sqrt{\sin(2\theta)}\) for \(0 \leq \theta \leq \pi/2\), thus the area of both loops (by symmetry) is

\[
A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta = \int_0^{\pi/2} (\sqrt{\sin(2\theta)})^2 \, d\theta = \frac{1}{2} \int_0^{\pi/2} \sin(2\theta) \, d\theta = -\cos(2\theta)
\]

\[
\bigg|_0^{\pi/2} = -\frac{1}{2} \left[ -1 - 1 \right] = 1
\]
(c) The total length equals twice the length of the curve in the first quadrant.

\[
L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = 2 \int_{0}^{\pi/2} \sqrt{\sin(2\theta) + \frac{\cos^2(2\theta)}{\sin(2\theta)}} d\theta
\]

4. (17 points) Consider the region in the first quadrant bounded by the coordinate axes, the curve \( y = \frac{2}{1 + x^2} \), and the line \( x = 1 \).

(a) Find the volume of the solid generated by revolving this region about the \( x \)-axis.

(b) Set up the integral to find the area of the surface generated by rotating this curve, \( 0 \leq x \leq 1 \), about the line \( y = 4 \).

Solution:

(a) By the Washer Method, we have

\[
V = \int_{0}^{1} \pi \left(\frac{2}{1 + x^2}\right)^2 dx = 4\pi \int_{0}^{1} \frac{1}{(x^2 + 1)^2} dx.
\]

To evaluate this integral, we use the trig substitution: \( x = \tan \theta \) and \( dx = \sec^2 \theta d\theta \). Then, \( x^2 + 1 = \sec^2 \theta \) and we obtain:

\[
V = 4\pi \int_{0}^{\pi/2} \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = 4\pi \int_{0}^{\pi/2} \frac{\cos^2 \theta}{\sin^2 \theta} d\theta
\]

\[
= 2\pi \int_{0}^{\pi/2} (1 + \cos 2\theta) d\theta = 2\pi \left[ \theta + \frac{\sin 2\theta}{2} \right]_{0}^{\pi/2} = \pi \left( \frac{\pi}{2} + 1 \right)
\]

Alternate Solution:

By the Shell Method, we have \( x = \sqrt{\frac{2 - y}{y}} = \sqrt{\frac{2}{y} - 1} \) and

\[
V = \int_{0}^{1} 2\pi y dy + \int_{1}^{2} 2\pi y \sqrt{\frac{2}{y} - 1} dy
\]

\[
= \pi y^2 \bigg|_{0}^{1} + \int_{1}^{2} 2\pi \sqrt{2y - y^2} dy = \pi + \int_{1}^{2} 2\pi \sqrt{1 - (1 - y)^2} dy.
\]

Let \( 1 - y = \sin \theta \), \( dy = -\cos \theta d\theta \),

\[
= \pi - \int_{0}^{-\pi/2} 2\pi \sqrt{1 - \sin^2 \theta}(-\cos \theta) d\theta = \pi - \int_{0}^{-\pi/2} 2\pi \cos^2 \theta d\theta
\]

\[
= \pi - \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_{0}^{-\pi/2} = \pi + \frac{\pi^2}{2}.
\]

(b) \( S = \int_{a}^{b} 2\pi r ds \) where \( ds = \sqrt{1 + (dy/dx)^2} dx \).

\[
\frac{dy}{dx} = -\frac{4x}{(1 + x^2)^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{16x^2}{(1 + x^2)^4} \Rightarrow S = \int_{0}^{1} 2\pi \left( 4 - \frac{2}{1 + x^2} \right) \sqrt{1 + \frac{16x^2}{(1 + x^2)^4}} dx
\]
Alternate Solution:

\[
\frac{dx}{dy} = \frac{-1}{y^2 \sqrt{2/y - 1}} \Rightarrow \left( \frac{dx}{dy} \right)^2 = \frac{1}{2y^3 - y^4} \Rightarrow S = \int_1^2 2\pi (4 - y) \sqrt{1 + \frac{1}{2y^3 - y^4}} \, dy
\]

5. (20 points) Consider the series \( \sum_{n=1}^{\infty} a_n \) where \( a_n = \frac{2}{(n + 1)(n + 3)} \). Let the partial sum \( s_n = \sum_{i=1}^{n} a_i \).

(a) Write the partial fraction decomposition of \( a_n \).
(b) Find a simple expression for \( s_n \).
(c) Is \( \{s_n\} \) monotonic? Justify your answer.
(d) Is \( \{s_n\} \) bounded? If so, find upper and lower bounds for \( s_n \).
(e) Does the given series converge? If so, what does it converge to?

Solution:

(a) \( a_n = \frac{1}{n + 1} - \frac{1}{n + 3} \)

(b) \( s_n = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n + 2} \right) + \left( \frac{1}{n + 1} - \frac{1}{n + 3} \right) \)

\( s_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n + 2} - \frac{1}{n + 3} = \frac{5}{6} - \frac{1}{n + 2} - \frac{1}{n + 3} \)

(c) The \( a_n \) terms are all positive so \( s_n \) is an increasing sequence and therefore monotonic.

(d) \( s_n \) is bounded below by \( a_1 = 1/4 \) and bounded above by

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{5}{6} - \frac{1}{n + 2} - \frac{1}{n + 3} \right) = \frac{5}{6}.
\]

(e) The series converges to the limit of the partial sums, or \( 5/6 \).

6. (15 points) Consider the series \( \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{2n + 1} x^{2n+1} \).

(a) For what values of \( x \) is the series absolutely convergent? conditionally convergent? divergent? Show all work and name any test(s) that you use.

(b) What is the sum of the series?

Solution:

(a) Use the Ratio Test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{3^{2n+3} x^{2n+3}}{2n + 3} \cdot \frac{2n + 1}{3^{2n+1} x^{2n+1}} \right| = \left| \frac{3^{2n+3}}{3^{2n+1}} \cdot \frac{2n + 1}{2n + 3} \cdot \frac{x^{2n+3}}{x^{2n+1}} \right| = \left| 3^2 x^2 \right| = |9x^2|
\]

The series converges when \( |9x^2| < 1 \) or \( |x| < 1/3 \).

(Alternatively, note that the series corresponds to the Maclaurin series for \( \arctan(3x) \). Since the series for \( \arctan(x) \) converges for \( |x| < 1 \), the series for \( \arctan(3x) \) converges for \( |3x| < 1 \) or \( |x| < 1/3 \).)
Check the endpoints. When \( x = 1/3 \), \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \) converges by the Alternating Series Test because

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{2n + 1} = 0 \text{ and } b_{n+1} = \frac{1}{2n + 3} < \frac{1}{2n + 1} = b_n. \text{ Similarly, when } x = -1/3, \\
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n + 1} \text{ also converges by the Alternating Series Test. Because } \sum_{n=0}^{\infty} \frac{1}{2n + 1} \text{ is divergent, we conclude that the series is conditionally convergent at } x = -1/3, 1/3, \text{ absolutely convergent on } (-1/3, 1/3) \text{ and divergent for all other values of } x.
\]

(b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{2n + 1} = \arctan(3x) \)

7. (25 points) Let \( g(x) = \sin(2x) \).

(a) State the definition of a Taylor series for an arbitrary function \( f \) centered at \( x = a \).
(b) Use the definition in part (a) to find a power series representation for \( g \) centered at \( a = \frac{\pi}{4} \).
(c) Find an error bound if \( T_3 \left( \frac{3\pi}{4} \right) \) is used to approximate \( \sin \left( \frac{3\pi}{2} \right) \).
(d) Find the sum of the series \( \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)! 2^{2n}} \).

Solution:

(a) \( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \).

(b) \( g(x) = \sin(2x) \), \( g''(x) = -2^3 \cos(2x) \), \( g(\frac{\pi}{4}) = 1 \), \( g''\left( \frac{\pi}{4} \right) = 0 \)
\( g'(x) = 2 \cos(2x) \), \( g(\frac{\pi}{4}) = 0 \), \( g'(\frac{\pi}{4}) = 2^4 \)
\( g''(x) = -2^2 \sin(2x) \), \( g(\frac{\pi}{4}) = 0 \), \( g''(\frac{\pi}{4}) = -2^2 \)
\( g(\frac{\pi}{4}) = 1 - \frac{2^2}{2!} \left( x - \frac{\pi}{4} \right)^2 + \frac{2^4}{4!} \left( x - \frac{\pi}{4} \right)^4 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left( x - \frac{\pi}{4} \right)^{2n} \).

(c) Note that \( |g^{(n)}(x)| \leq 2^n \) for all \( x \). By Taylor’s Formula,
\[
R_3(x) = \frac{g^{(4)}(z)}{4!} (x - a)^4 \Rightarrow R_3 \left( \frac{3\pi}{4} \right) = \frac{g^{(4)}(z)}{4!} \left( \frac{3\pi}{4} - \frac{\pi}{4} \right)^4 \leq \frac{2^4}{4!} \left( \frac{\pi}{2} \right)^4 = \frac{\pi^4}{4!}.
\]

Alternate Solution:

Use the Alternating Series Estimation Theorem. The series
\[
g \left( \frac{3\pi}{4} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} \left( \frac{3\pi}{4} - \frac{\pi}{4} \right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \cdots
\]
satisfies
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\pi^{2n}}{(2n)!} = 0 \text{ because } \lim_{n \to \infty} \frac{x^n}{n!} = 0,
\]
and
\[
b_{n+1} = \frac{\pi^{2n+2}}{(2n + 2)!} = \frac{\pi^2}{(2n + 2)(2n + 1)} \cdot \frac{\pi^{2n}}{(2n)!} = \frac{\pi^2}{(2n + 2)(2n + 1)} \cdot b_n < b_n \text{ for } n \geq 1.
\]
It follows that the estimation error

\[ |\sin \left( \frac{3\pi}{2} \right) - T_3 \left( \frac{3\pi}{4} \right)| = |\sin \left( \frac{3\pi}{2} \right) - \left( 1 - \frac{\pi^2}{2!} \right)| \leq \frac{\pi^4}{4!}. \]

(d) \[ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)! 2^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left( \frac{\pi}{4} \right)^{2n} = \sin(2x) \text{ for } x = 0 \text{ or } x = \pi/2, \text{ hence the sum of the series equals } \sin 0 = \sin \pi = 0. \]

Alternate Solution:
The series corresponds to the Maclaurin series for \( \cos x \) with \( x = \pi/2 \). The sum of the series is \( \cos(\pi/2) = 0 \).